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Gamow Vectors and Time Asymmetry

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We prove that Gamow vectors are important tools in the quantum theory of irreversibility. We use the mathematical formalism of rigged Hilbert spaces. We discuss some spectral formulas that include Gamow vectors as well as some results concerning Gamow vectors. The role of the time-reversal operator is studied. The formalism can be applied to formulate a sense of irreversibility in cosmology.

1. INTRODUCTION

This is a paper on time asymmetry, cosmology, and quantum mechanics. Time asymmetry is usually explained in two different ways. The first one and most popular lies at the level of statistical mechanics. A *coarse graining* is introduced, some approximations are usually made, and a master equation is obtained [1–3]. However, this method is somehow arbitrary since there is no general rule to define which should be considered as the relevant system and which should be considered as the irrelevant bath or reservoir.

Our intention is to discuss some particular aspects of a formalism which is, in principle, free of this ambiguity. This formalism has its origin in some ideas due to I. Prigogine and A. Bohm, who have worked independently [4–10]. In particular, I. Prigogine and coworkers use the term *intrinsic irreversibility* in this context to underline that this formalism describes irreversibility in physics at a fundamental level.

The idea behind the formulation of intrinsic irreversibility lies in the fact that the group giving the time evolution of certain systems with reversible equations of motion (for instance, e^{-itH} for nonrelativistic quantum systems or U^n for chaotic maps, where U is the Frobenius–Perron operator, etc.)

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splits, in the presence of resonances into two semigroups that act in different spaces. One of these two spaces accounts for the processes in the past, the other for the future processes. This implies the existence of a time arrow which points to the past in the past space and to the future in the future space. This means that time symmetry can be broken in intrinsically irreversible systems and, hence, a time asymmetry can be introduced canonically.

In the case of nonrelativistic quantum systems with resonances, the spaces of pure quantum states for the past and the future are not represented as Hilbert spaces, but instead we equip the original Hilbert space of states with new spaces. One of these contains the past processes, the other the future processes. These are time symmetric with regard to one another and are related through a time-reversal operator. Each would therefore correspond to an arrow of time.

The choice of the new spaces and, hence, of the arrow of time can be based on cosmological considerations. In fact, if we assume that the universe is a time-orientable manifold in which we have chosen an orientation (that of the future), this choice will single out one of these two possible spaces.

The choice of an arrow of time can also be illustrated by means of the use of Reichenbach diagrams [11, 45]. We can consider all regular states as outgoing states in a Reichenbach diagram. These outgoing states would live in the new states corresponding to the future.

In order to illustrate the formalism of time asymmetry, we focus on nonrelativistic quantum systems with resonances in order to obtain consequences of our analysis relavant to cosmology. In this case, we have a Hilbert state of pure states $\mathcal{H}$ and we equip it with a dual pair. A dual pair is formed by two vector spaces: a given topological vector space $\mathbf{\Phi}$ and its dual $\mathbf{\Phi}^{\times}$ (the vector space of continuous antilinear functionals on $\mathbf{\Phi}$), such that the following relation holds:

$$\mathbf{\Phi} \subset \mathcal{H} \subset \mathbf{\Phi}^{\times} \tag{1.1}$$

Such spaces are called rigged Hilbert spaces (RHS) and their properties are summarized in Appendix A. We shall use the following notation for the space corresponding to the past

$$\mathbf{\Phi}^{+} \subset \mathcal{H} \subset (\mathbf{\Phi}^{+})^{\times} \tag{1.2}$$

and for the space corresponding to the future

$$\mathbf{\Phi}^{-} \subset \mathcal{H} \subset (\mathbf{\Phi}^{-})^{\times} \tag{1.3}$$

Quantum resonance phenomena illustrate the need for this double equipping of the Hilbert space of states. In fact, resonance scattering represents two processes. One is the creation of the resonance (growing or capture process), which takes place in (1.2), the decay in (1.3). The need for these extensions of the Hilbert space can be made obvious if we introduce the Gamow vectors or vector states for the exponentially growing or decaying part of the resonance. As is well known, these Gamow vectors cannot be represented by a well-behaved square-integrable wave function, but can be defined as proper functionals on the spaces $\Phi^{\pm}$ and hence as elements in $(\Phi^{\pm})^{\times}$. In the present paper, we shall introduce a new type of Gamow vector and discuss the role played by Gamow vectors in time-asymmetric quantum mechanics and cosmology.

Of the tools we shall use, two are of particular importance: analytic continuation of the wave functions in the energy representation, making use of Hardy functions on a half-plane of the complex plane $\mathbb{C}$, and rigged Hilbert spaces [8–12]. We have already shown that rigged Hilbert spaces are used in order to equip the Hilbert space of states. Due to causality conditions [13], the extensions, are constructed using Hardy functions, which are defined in Appendix B, where we also give some of their most interesting properties.

We present a formulation of decay phenomena such that the spaces $\Phi^{\pm}$ (which represent the closure of the linear spaces of physically realizable state vectors with respect to a topology stronger than the Hilbert space topology) can be represented by means of spaces of *entire* analytic functions in the energy representation. This choice is based on the following ideas:

1. We would like to represent the spaces $\Phi^{\pm}$ on spaces of functions having the maximun possible analyticity in order to use contour integrations in our calculations.

2. The kind of structures under study and, in particular, our new Gamow vectors require larger spaces $(\Phi^{\pm})^{\times}$. This can be obtained with smaller $\Phi^{\pm}$. It is necessary, however, to remark that the functions in the representation spaces for $\Phi^{\pm}$ also must be Hardy on the upper half-plane (for Φ^{-}) and the lower half-plane (for Φ^{-}) in order to fulfill certain causality conditions [13].

One of the objectives of this formulation is to show how the presence of poles of the analytic continuation to the second sheet of the Riemann surface, associated with the transformation $E = p^2$, of the S-matrix in the energy representation allows us to find the manifestation of the existence of an arrow of time. Under certain general conditions in the classical case, these poles can be related to resonances [8, 9].

We also discuss a generalization of quantum mechanics which preserves as much as possible the features of the standard formalism. In particular, we use the notions of generalized state vectors and their brackets. In particular the latter has been absent in previous treatments [8–10].

The general mathematical framework for such a description is the theory of rigged Hilbert spaces (RHS). RHS were invented by Gel'fand [14] and its properties studied by Gel'fand [14], Maurin [15, 16], and others [17, 18]. Its use for a rigorous implementation of the Dirac formulation of quantum mechanics has been proposed by Antoine [19], Bohm [20, 21], Roberts [22], and Melsheimer [18]. Rigged Hilbert spaces of analytic functions have also been used to describe decay and resonant behavior [23, 9, 10]. Gamow vectors, defined as exponentially decaying and growing state vectors, can be rigorously defined in this context [8, 10, 24]. Then the RHS description of decay serves as a mathematical basis to show an arrow of time [8, 25]. Thus, one could find the basis of a quantum theory of irreversibility on this description. Nevertheless, one should note that the mathematical theory of RHS does not rigorously describe certain particular aspects of the formalism we want to discuss. In fact, certain objects that we wish to introduce, like the "norm" of the Gamow vectors as well as certain "scalar" products and "Dirac deltas," have not been rigorously defined, since they are essentially products of distributions. Thus, one of our objectives is to show the need for completing the mathematical theory of operators on RHS.

The definition and main properties of RHS (also called Gel'fand triplets), including the important Nuclear Spectral Theorem or Gel'fand–Maurin theorem, which is the basis for the RHS implementation of the Dirac formalism, are presented in Appendix A.

We begin with a Hilbert space $\mathcal{H}$ and a self-adjoint operator H on $\mathcal{H}$, which represents the Hamiltonian of a given physical system. Usually, the Hamiltonian can be decomposed into a "free" part H_0 plus an interaction, which is usually given by a potential V, so that

$$H = H_0 + V \tag{1.4}$$

In order to simplify the problem under discussion, we make the following assumptions.

1. The free Hamiltonian H_0 is unitarily equivalent to the multiplication operator $\mathcal{E}$ on $L^2(\mathbb{R}^+)$ (this is the space of square-integrable functions with respect to the Lebesgue measure on the positive part of the real line):

$$\mathcal{E}f(\omega) = \omega f(\omega) \qquad \text{for all} \quad \omega \in \mathbb{R}^+ \tag{1.5}$$

In particular, this means that H_0 has continuous spectrum, $\sigma(H_0) = \mathbb{R}^+ = [0, \infty)$ only, and is *not degenerate*. This nondegeneracy hypothesis is not essential here, but introduce it for simplicity in the notation. Dropping this hypothesis means that $f(\omega)$ in (1.5) depends on some other variables.

2. The Møller wave operators $\Omega_\pm$, also called scattering operators, exist. Furthermore, we assume *asymptotic completeness* [26, 27]. Thus, H and H_0 are related by $H = \Omega_\pm H_0 \Omega_\pm^\dagger$. As a consequence, the continuous spectrum of H is $\mathbb{R}^+ = [0, \infty)$ and is not degenerate. Also, the S operator (S matrix) exists and is unitary.

3. The S operator in the energy representation, $S(\omega)$, has an analytic continuation to the two-sheeted Riemann surface associated with the transfor-

mation $w = \eta^2$. We also assume that the analytic continuation of $S(\omega)$ to the second sheet, given by $S_{\text{II}}(z)$, has as only singularities a *finite* number of pairs of simple poles, located at the points $z_j = \omega_j - i\Gamma_j/2$ and $z_j^* = \omega_j + i\Gamma_j/2$ with $\omega_j > 0$, $\gamma_j > 0$, $j = 1, 2, \ldots, N$. Eventually, one may also have simple poles on the negative part of the real axis (second sheet). One can always construct a one-dimensional model with these properties by using the Gel'fand–Levitan theory [28]. For the sake of simplicity, we will assume the presence of only one pair of poles located $z_R = \omega_R \pm i\Gamma_R/2$ with $\omega_R > 0$ and $\Gamma_R > 0$. The function $S_{\text{II}}(\omega)$ is *polynomially bounded* at infinity. This happens in some realistic models [29] having spherically symmetric, finite-range potentials. The cut of the Riemann surface has two rims. On each of the rims the values of the function $S(\omega)$ are well defined as the limits of $S(z)$ as $\text{Im}\, z \to 0$ and $\text{Re}\, z > 0$ from the upper half-plane and lower half-plane, respectively (first sheet). We denote these values as $S(\omega \pm i0)$.

2. ANALYTIC CONTINUATIONS

Let us consider the two rigged Hilbert spaces

$$\Delta_\pm \subset \mathcal{H}^2_\pm \subset \Delta^\times_\pm \tag{2.1}$$

as defined in Appendix B. By $\mathcal{H}^2_\pm$ we denote the spaces of Hardy functions on the $\left\{ \substack{\text{upper} \\ \text{lower}} \right\}$ half-plane. The functions in $\Delta_\pm$ are entire analytic on each sheet of the Riemann surface associated with the transformation $\omega = \eta^2$ (and on the whole Riemann surface itself, the function on one sheet being *identical* to the function on the other), where ω represents the energy (see Appendix B). In addition, they are Hardy functions on the $\left\{ \substack{\text{upper} \\ \text{lower}} \right\}$ half-plane on *both sheets*. Since we are only interested in the behavior of our functions on the second sheet, we will work on this sheet only.

The values that any function on $\mathcal{H}^2_\pm$ takes on the real axis determine all the values of the function. The same property is true for the values on the positive part of the real axis $\mathbb{R}^+$, after the van Winter theorem [30]. Thus, if we define the following mappings for any $f_\pm \in \mathcal{H}^2_\pm$

$$\theta_\pm\colon \quad f_\pm \mapsto f_\pm\big|_{\mathbb{R}^+} \tag{2.2}$$

(the latter symbol means the restriction of these functions to $\mathbb{R}^+$), they are *one to one* onto linear mappings. Let us call $\Gamma_\pm$ the *images* of $\Delta_\pm$ by $\theta_\pm$. One can transport the topological properties from $\Delta_\pm$ to $\Gamma_\pm$ with $\theta_\pm$ and show that the triplets

$$\Gamma_\pm \subset L^2(\mathbb{R}^+) \subset \Gamma^\times_\pm \tag{2.3}$$

are rigged Hilbert spaces.

According to our assumptions, H_0 is unitarily equivalent to the operator multiplication $\mathscr{E}$ on $L^2(\mathbb{R}^+)$. It is rather simple to show that if $f_\pm(\omega) \in \Gamma_\pm$, then $\mathscr{E} f_\pm(\omega) = \omega f_\pm(\omega) \in \Gamma_\pm$, so that the operator $\mathscr{E}$ leaves $\Gamma_\pm$ invariant. One can also show that $\mathscr{E}$ is a continuous mapping from $\Gamma_\pm$ into themselves.

In the preceding section, we assumed that H_0 and $\mathscr{E}$ are unitarily equivalent. Then, there must exist a unitary operator U: $\mathscr{H} \mapsto L^2(\mathbb{R}^+)$ with $H_0 = U^{-1}\mathscr{E}U$. Next, we define the following spaces:

1. $\Phi_\pm = U^{-1}\Gamma_\mp$. Since U^{-1} is a one-to-one mapping from $\Gamma_\mp$ onto $\Phi_\pm$, it can transport the topological properties from $\Gamma_\mp$ to $\Phi_\pm$. Thus,

$$\Phi_\pm \subset \mathscr{H} \subset \Phi_\pm^\times \tag{2.4}$$

are rigged Hilbert spaces.

2. $\Phi^\pm = \Omega_\pm \Phi_\pm$. This provides a new pair of RHS:

$$\Phi^\pm \subset \mathscr{H}_{ac} \subset (\Phi^\pm)^\times \tag{2.5}$$

Here, $\mathscr{H}_{ac}$ is the *absolutely continuous Hilbert space* of the total Hamiltonian H. This is defined as the orthogonal complement of the Hilbert space spanned by the bound states of H (we assume the absence of continuous singular spectrum). According to our hypothesis, the restriction of H to $\mathscr{H}_{ac}$ has nondegenerate continuous spectrum given by $\mathbb{R}^+$. We recall that, according to our hypothesis of asymptotic completeness, we have $\mathscr{H}_{ac} = \Omega_+\mathscr{H} = \Omega_-\mathscr{H}$.

The operators H_0 and H leave invariant the spaces $\Phi_\pm$ and $\Phi^\pm$, respectively. Furthermore, they are continuous on the spaces they leave invariant. The second version (see Appendix A) of the *Gel'fand–Maurin theorem* says that there exist complete sets of generalized eigenvectors $|\omega_\pm\rangle \in \Phi_\pm^\times$ of H_0 and $|\omega^\pm\rangle \in (\Phi^\pm)^\times$ of H, where ω runs out $\mathbb{R}^+$. One can also find the relation [10]

$$|\omega^\pm\rangle = \Omega_\pm|\omega_\pm\rangle; \qquad \omega \in \mathbb{R}^+ \tag{2.6}$$

which is known as the *Lippmann–Schwinger* equation. Certain *formal* expressions can also be found:

$$H_0 = \int_0^\infty \omega|\omega_\pm\rangle\langle\omega_\pm|\, d\omega; \qquad I = \int_0^\infty |\omega_\pm\rangle\langle\omega_\pm|\, d\omega \tag{2.7}$$

Rigorously speaking, this I is the identity mapping from $\Phi_\pm$ into $\Phi_\pm^\times$. Similar expressions are

$$H = \int_0^\infty \omega|\omega^\pm\rangle\langle\omega^\pm|\, d\omega; \qquad I = \int_0^\infty |\omega^\pm\rangle\langle\omega^\pm|\, d\omega \tag{2.8}$$

Here, I should be looked at as the identity mapping from $\Phi^\pm$ into $(\Phi^\pm)^\times$.

Since $|\omega^{\pm}\rangle$ are generalized bases for $\Phi^{\pm}$ as $\omega \in \mathbb{R}^{+}$, being given a vector $\phi^{\pm} \in \Phi^{\pm}$, one can write

$$\phi^{\pm} = |\phi^{\pm}\rangle = \int_0^{\infty} \langle\omega^{\pm}|\phi^{\pm}\rangle|\omega^{\pm}\rangle \, d\omega = \int_0^{\infty} \phi^{\pm}(\omega)|\omega^{\pm}\rangle \, d\omega \tag{2.9}$$

One can show [11] that the wave function $\phi^{\pm}(\omega)$ belong to $\Gamma_{\pm}$, respectively. We recall that $\langle\omega^{\pm}|\phi^{\pm}\rangle = \langle\phi^{\pm}|\omega^{\pm}\rangle^{*} = \phi^{\pm}(\omega)$. Similar expressions are valid for $\varphi_{\pm} = |\varphi_{\pm}\rangle \in \Phi_{\pm}$. The next three formulas are valid with the plus and minus signs either as superindices or as subindices and therefore we omit them. The square of the norm of some vector $|\phi\rangle$ (in $\Phi^{\pm}$ or in $\Phi_{\pm}$) is given by

$$\langle\phi|\phi\rangle = \int_0^{\infty} \langle\phi|\omega\rangle\langle\omega|\phi\rangle \, d\omega = \int_0^{\infty} |\phi(\omega)|^2 \, d\omega \tag{2.10}$$

We can also use the bra notation:

$$\langle\psi| = \int_0^{\infty} \langle\psi|\omega\rangle\langle\omega| \, d\omega = \int_0^{\infty} \psi^{*}(\omega)\langle\omega| \, d\omega \tag{2.11}$$

and

$$\langle\psi|\psi\rangle = \int_0^{\infty} \langle\psi|\omega\rangle\langle\omega|\psi\rangle \, d\omega = \int_0^{\infty} |\psi^{*}(\omega)|^2 \, d\omega \tag{2.12}$$

Warning. Note that the above formulas can be written for the elements of $\Phi_{\pm}$ or $\Phi^{\pm}$ and not for all the elements of $\mathcal{H}$ or $\mathcal{H}_{ac}$ as currently assumed. This is a consequence of the Gel'fand–Maurin theorem. For different reasons, the expression

$$|\omega_{+}\rangle = S(\omega)|\omega_{-}\rangle \tag{2.13}$$

is not true in the present context. We will discuss this point later.

Now, let $\phi^{\pm} \in \Phi^{\pm}$. The function on $\mathbb{R}^{+}$ given by $\phi^{\pm}(\omega) = \langle\omega^{\pm}|\phi^{\pm}\rangle = \langle\phi^{\pm}|\omega^{\pm}\rangle^{*}$ belongs to $\Gamma_{\pm}$. This means that $\phi^{\pm}(\omega)$ admits an analytic continuation to an *entire function* which is, in addition, a Hardy function on the $\left\{\begin{smallmatrix}\text{upper}\\ \text{lower}\end{smallmatrix}\right\}$ half-plane. Let z be any complex number. Then, the mapping

$$F_{\pm}: \quad \Phi^{\pm} \mapsto \mathbb{C} \quad \text{such that} \quad F_{\pm}(\phi^{\pm}) = [\phi^{\pm}(z^{*})]^{*} \tag{2.14}$$

is antilinear and continuous on $\Phi^{\pm}$ and hence it belongs to $(\Phi^{\pm})^{\times}$. Antilinearity is obvious. Continuity is proven in Appendix C. In the Dirac notation, we write $F_{\pm}(\phi^{\pm}) = \langle\phi^{\pm}|z^{\pm}\rangle = \langle z^{\pm}|\phi^{\pm}\rangle^{*}$, so that $|z^{\pm}\rangle$ and $F_{\pm}$ represent the same functional. The same kind of considerations can be made for $\phi_{\pm} \in \Phi_{\pm}$. For $\phi^{\pm}$, one has

$$[\phi^{\pm}(z^*)]^* = \langle \phi^{\pm} | z^{\pm} \rangle \Leftrightarrow \phi^{\pm}(z^*) = \langle z^{\pm} | \phi^{\pm} \rangle \Leftrightarrow \phi^{\pm}(z) = \langle z^{\pm} | \phi^{*\pm} \rangle \tag{2.15}$$

Now, let us take φ and ϕ both in the same of any of the spaces $\Phi_{\pm}$ or $\Phi^{\pm}$. Let us omit for a moment the $\pm$ signs for simplicity. Consider the following scalar product:

$$\langle \varphi | \phi \rangle = \int_0^{\infty} \varphi^*(\omega)\, \phi(\omega)\, d\omega \tag{2.16}$$

If $\varphi(\omega) \in \Gamma_{\pm}$, then $\varphi^*(\omega) \in \Gamma_{\mp}$. However, if $\varphi(z)$ is analytic, $\varphi^*(z)$ is not. The analytic continuation of $\varphi^*(\omega)$ is given by $\varphi^*(z^*)$, sometimes also called $\varphi^{\#}(z)$. Obviously, $\varphi^{\#}(\omega) = \varphi^*(\omega)$ if ω is real and the above scalar product can be written as

$$\langle \varphi | \phi \rangle = \int_0^{\infty} \varphi^{\#}(\omega)\, \phi(\omega)\, d\omega \tag{2.17}$$

The path of integration can now be deformed (see Fig. 1) to Γ so that

$$\langle \varphi | \phi \rangle = \int_{\Gamma} \varphi^{\#}(z)\, \phi(z)\, dz = \int_{\Gamma} \langle \varphi | z \rangle \langle z^* | \phi \rangle\, dz \tag{2.18}$$

The second identity in (2.18) comes from (2.15). By analogy with (2.7) and (2.8), we can now write the following formula (now recovering the indices $\pm$):

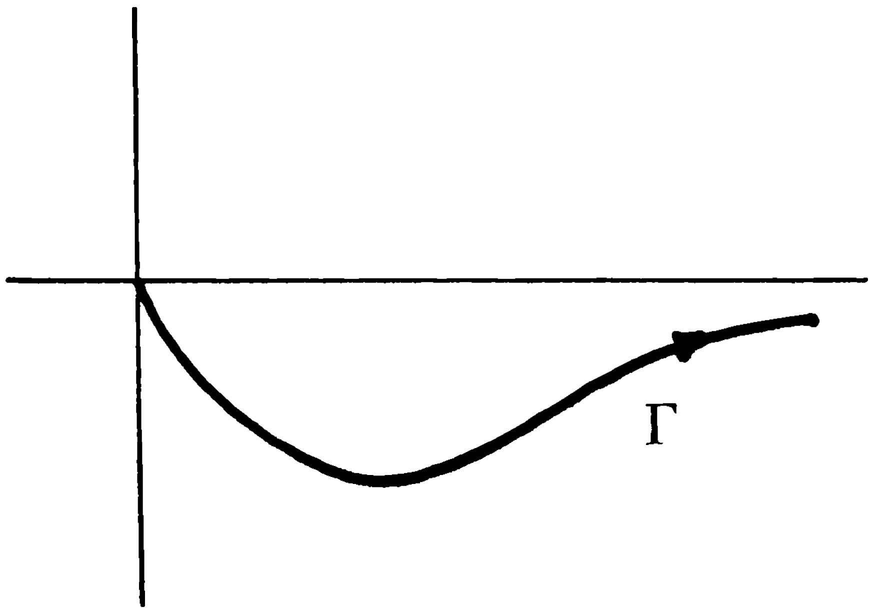

Fig. 1.

$$I = \int_\Gamma |z^\pm\rangle\langle z^{*\pm}| \, dz \tag{2.19}$$

As we will prove in the next section, $H|z^\pm\rangle = z|z^\pm\rangle$ and $\langle z^\pm|H = z^*\langle z^\pm|$. Thus, formula (2.19) gives

$$\langle \varphi^\pm|H\phi^\pm\rangle = \int_\Gamma \langle \varphi^\pm|z^\pm\rangle\langle z^{*\pm}|H\phi^\pm\rangle \, dz = \int_\Gamma z\langle \varphi^\pm|z^\pm\rangle\langle z^{*\pm}|\phi^\pm\rangle \, dz \tag{2.20}$$

This expression can be obtained in an equivalent way if we write (omitting again the indices $\pm$)

$$\langle \varphi|H\phi\rangle = \int_0^\infty \varphi^\#(\omega)(H\phi)(\omega)\, d\omega \tag{2.21}$$

$$= \int_0^\infty \langle \phi|\omega\rangle\langle\omega|H\phi\rangle\, d\omega = \int_0^\infty \omega\langle \phi|\omega\rangle\langle\omega|\phi\rangle\, d\omega \tag{2.22}$$

$$= \int_0^\infty \omega\varphi^\#(\omega)\phi(\omega)\, d\omega = \int_\Gamma z\varphi^\#(z)\phi(z)\, dz$$

$$= \int_\Gamma z\langle \varphi|z\rangle\langle z^*|\phi\rangle\, dz \tag{2.23}$$

and can be written as

$$H = \int_\Gamma z|z^\pm\rangle\langle z^{*\pm}| \, dz \tag{2.24}$$

Certain formal manipulations involving (2.19) and (2.24) are coherent with these results. For instance,

$$H \cdot I = \int_\Gamma H|z^\pm\rangle\langle z^{*\pm}| \, dz = \int_\Gamma z|z^\pm\rangle\langle z^{*\pm}| \, dz \tag{2.25}$$

$$I \cdot H = \int_\Gamma |z^\pm\rangle\langle z^{*\pm}|H \, dz = \int_\Gamma |z^\pm\rangle\langle z^{*\pm}|z \, dz \tag{2.26}$$

Some kinds of formal scalar products between generalized eigenvectors of H are meaningful in a distributional sense. For instance, if we multiply the second integral in (2.7) to the left by $\langle\phi^\pm|$ and to the right by $|\omega'^\pm\rangle$, we get

$$\langle\phi^\pm|\omega'^\pm\rangle = \int_0^\infty \langle\phi^\pm|\omega^\pm\rangle\langle\omega^\pm|\omega'^\pm\rangle\, d\omega \Rightarrow \langle\omega^\pm|\omega'^\pm\rangle = \delta(\omega - \omega') \tag{2.27}$$

This delta is well defined as some kind of distribution. Proceeding by analogy, if $z' \in \Gamma$, we have

$$\langle \phi^{\pm}|z'^{\pm}\rangle = \int_{\Gamma} \langle \phi^{\pm}|z^{\pm}\rangle\langle z^{*\pm}|z'^{\pm}\rangle \, dz \tag{2.28}$$

As a result, one has

$$\langle z^{*\pm}|z'^{\pm}\rangle = \delta_{\Gamma}(z - z') \tag{2.29}$$

and formula (2.28) can be looked at as the definition of the delta function in (2.29).

3. BEHAVIOR OF $|z^{\pm}\rangle$ AND EVOLUTION THEOREMS

The purpose of this section is to study the time evolution of the generalized vectors $|z^{\pm}\rangle$. This study has already been made in a situation in which the wave functions $\phi^{\pm}(\omega)$ of the vectors $\phi^{\pm} \in \Phi^{\pm}$ admit analytic continuation on a half-plane only [11]. Now, we will work with entire functions, which will produce an *enriched mathematical structure*.

First of all, we want to show that

$$H|z^{\pm}\rangle = z|z^{\pm}\rangle \tag{3.1}$$

i.e., $|z^{\pm}\rangle$ is a *right eigenvector* of H with eigenvalue z.

In order to obtain this result, we have to use the definition of generalized eigenvector given in Appendix A. According to this definition, we have to show that

$$\langle H\phi^{\pm}|z^{\pm}\rangle = z\langle \phi^{\pm}|z^{\pm}\rangle, \qquad \forall \phi^{\pm} \in \Phi^{\pm} \tag{3.2}$$

The proof of (3.2) is simple. Take an arbitrary $\omega \in \mathbb{R}^{+}$. Then, the Gel'fand–Maurin theorem gives $H|\omega^{\pm}\rangle = \omega|\omega^{\pm}\rangle$ (see Appendix A). This means that, for any $\phi^{\pm} \in \Phi^{\pm}$, one has

$$\langle H\phi^{\pm}|\omega^{\pm}\rangle = \omega\langle \phi^{\pm}|\omega^{\pm}\rangle = \omega[\phi^{\pm}(\omega)]^{*} \tag{3.3}$$

The analytic continuation of the functions in (3.3) gives, at any complex z, the following value:

$$\langle H\phi^{\pm}|z^{\pm}\rangle = z[\phi^{\pm}(z^{*})]^{*} = z\langle \phi^{\pm}|z^{\pm}\rangle \tag{3.4}$$

which proves (3.2).

We can also study the time evolution of the generalized eigevectors $|z^{\pm}\rangle$. In order to do it, we need to recall how to construct the extensions to the duals of non-self-adjoint operators on a rigged Hilbert space. Let $\Phi \subset \mathcal{H} \subset \Phi^{\times}$ be a rigged Hilbert space and U an operator on $\mathcal{H}$ fulfilling the following properties:

1. The Hilbert space adjoint adjoint of U, $U^{\dagger}$, leaves Φ invariant, i.e., $U^{\dagger}\Phi \subset \Phi$.

2. $U^\dagger$ is continuous on Φ.

Then, there is a unique extension of U by continuty to $\Phi^\times$. This is defined by

$$\langle U^\dagger \varphi | F\rangle = \langle \varphi | UF\rangle, \qquad \forall \varphi \in \Phi \quad \text{and all } F \in \Phi^\times \tag{3.5}$$

Therefore, in order to find the time evolution of $|z^\pm\rangle$, we have first to the following:

(i) For $t > 0$, e^{itH} leaves the space Φ^- invariant and is continuous on it. On the other hand, for any $t < 0$, e^{itH} does not leave Φ^- invariant. Note that e^{itH} is the Hilbert space adjoint of the evolution operator e^{-itH}. For this reason, e^{-itH} is well defined in the dual $(\Phi^-)^\times$ for $t > 0$ and not for $t < 0$.

(ii) For $t < 0$, e^{itH} leaves invariant Φ^+ and is continuous on it. For any $t > 0$, e^{itH} does not leave Φ^+ invariant. Therefore, e^{-itH} is well defined on $(\Phi^+)^\times$ for $t < 0$ and not for $t > 0$.

Proofs of (i) and (ii) are rather technical and not very different from the proofs for the similar situation described in ref. 10 p. 80.

Thus, the usual group giving the time evolution of a quantum state e^{-itH} splits into two semigroups:

1. e^{-itH} acting on $(\Phi^-)^\times$ for $t > 0$ only.
2. e^{-itH} acting on $(\Phi^+)^\times$ for $t < 0$ only.

Note that for $t > 0$, e^{-itH} acts on Φ^- *as a subspace* of $(\Phi^-)^\times$, but does not leave it invariant. As a matter of fact, $e^{-itH}\,\Phi^- \subset (\Phi^-)^\times$. Analogously, for $t < 0$, $e^{-itH}\,\Phi^+ \subset (\Phi^+)^\times$, but e^{-itH} does not leave Φ^+ invariant.

In the next section, we shall discuss the need for this splitting in the presence of resonance poles and its consequences from the point of view of the theory of irreversibility.

Using (3.5), we obtain [11]

$$\text{for} \quad t > 0, \qquad \langle e^{itH}\phi^- | \omega^-\rangle = e^{-it\omega}\langle \phi^- | \omega^-\rangle \tag{3.6}$$

$$\text{for} \quad t < 0, \qquad \langle e^{itH}\phi^+ | \omega^+\rangle = e^{-it\omega}\langle \phi^+ | \omega^+\rangle \tag{3.7}$$

The expressions for $\langle e^{itH}\phi^\pm | z^\pm\rangle$ can be obtained through analytic continuation. Thus, we arrive at the obvious conclusion that

$$\text{for} \quad t > 0, \qquad \langle e^{itH}\phi^- | z^-\rangle = e^{-itz}\langle \phi^- | z^-\rangle \tag{3.8}$$

$$\text{for} \quad t < 0, \qquad \langle e^{itH}\phi^+ | z^+\rangle = e^{-itz}\langle \phi^+ | z^+\rangle \tag{3.9}$$

which implies that

$$\text{for} \quad t > 0, \qquad e^{-itH}|z^-\rangle = e^{-itz}|z^-\rangle \tag{3.10}$$

$$\text{for} \quad t < 0, \qquad e^{-itH}|z^+\rangle = e^{-itz}|z^+\rangle \tag{3.11}$$

These formulas have an obvious consequence, which shows the nature of these generalized states: if Im $z < 0$, $|z^-\rangle$ decays exponentially as t goes to infinity. However, if Im $z > 0$, $|z^-\rangle$ *grows exponentially* as $t \to \infty$ and therefore it is not defined in this limit. This strange behavior for $|z^-\rangle$ with Im $z > 0$ would lead us to assume, at first sight, that this mathematical object is physically anomalous and should not be considered. Nevertheless, we shall see that it may be an interesting object. Equation (3.9) says that $|z^+\rangle$ grows exponentially if Im $z < 0$ as $t \to -\infty$ and decays exponentially if Im $z > 0$ as $t \to -\infty$. The first case is physically relevant in the sense that it represents the exponentially growing part of the creation of a resonance or capture state. The second is called the anomalous decaying Gamow vector.

We can also write the complex conjugate of (3.2):

$$\langle H\phi^\pm | z^\pm\rangle^* = z^*\langle \phi^\pm | z^\pm\rangle^* = z^*\langle z^\pm | \phi^\pm\rangle \tag{3.12}$$

$$\langle H\phi^\pm | z^\pm\rangle^* = \langle z^\pm | H\phi^\pm\rangle = \langle z^\pm | H | \phi^\pm\rangle \tag{3.13}$$

Combining (3.12) with (3.13), we obtain the formula for the *left eigenvectors* of H with eiegenvalue z^*:

$$\langle z^\pm | H = z^*\langle z^\pm | \tag{3.14}$$

If we use the manipulations in (3.12) and (3.13) in (3.8) and (3.9), we also get the following result:

$$\text{for} \quad t > 0, \qquad \langle z^- | e^{itH} = e^{itz^*}\langle z^- | \tag{3.15}$$

$$\text{for} \quad t < 0, \qquad \langle z^+ | e^{itH} = e^{itz^*}\langle z^+ | \tag{3.16}$$

and therefore

$$\text{for} \quad t < 0, \qquad \langle z^- | e^{-itH} = e^{-itz^*}\langle z^- | \tag{3.17}$$

$$\text{for} \quad t > 0, \qquad \langle z^+ | e^{-itH} = e^{-itz^*}\langle z^+ | \tag{3.18}$$

We see that when Im $z \neq 0$, if the left eigenvector of H grows, the corresponding (right) eigenvector decays and vice versa. We will come back to this subject when we discuss the properties of the Gamow vectors in the next section.

Final Remarks. 1. We make a brief comment on the nature of the left eigenvectors of H. Let $\Phi \subset \mathcal{H} \subset \Phi^\times$ be a RHS and F an element of $\Phi^\times$. We can define a mapping F' from Φ into the set of complex numbers as follows:

$$F'(\varphi) = [F(\varphi)]^* \qquad \text{for all} \quad \varphi \in \Phi \tag{3.19}$$

It is obvious that F' is a continuous *linear* functional on Φ. Thus, (3.19) gives a one-to-one onto mapping between $\Phi^\times$ and the space Φ' of continuous linear functionals on Φ. Then, observe that $\langle z^{\pm}|\phi^{\pm}\rangle = \langle\phi^{\pm}|z^{\pm}\rangle^*$ is a particular case of (3.19) and, therefore, $\langle z^{\pm}|$ are the continuous *linear* functionals on $\Phi^{\pm}$ that correspond to the continuous *antilinear* functionals $|z^{\pm}\rangle$ through (3.19).

2. To end this section, we comment on the possible relation between $|z^{+}\rangle$ and $|z^{-}\rangle$. First, one sees that the intersection $(\Phi^{+})^{\times} \cap (\Phi^{-})^{\times}$ must contain *at least* the Hilbert space $\mathcal{H}$. One may ask the following question: since the actions of both $|z^{+}\rangle$ and $|z^{-}\rangle$ give on the vectors ϕ^{+} and ϕ^{-}, respectively, the value of the analytic continuation of their respective wave functions, given by $\langle\omega^{+}|\phi^{+}\rangle$ and $\langle\omega^{-}|\phi^{-}\rangle$, at z, is it possible to identify both functionals on $(\Phi^{+})^{\times} \cap (\Phi^{-})^{\times}$? The fact that they both are generalized eigenvectors of H with the same eigenvalue reinforces this possibility. Nevertheless, the answer is *no*, as we see here:

If $\phi^{\pm} \in \Phi^{\pm}$, its wave function in the energy representation belongs to $\Gamma_{\mp}$. This means that it has an analytic continuation to an entire function which is Hardy class on the $\left\{\begin{smallmatrix}\text{upper}\\ \text{lower}\end{smallmatrix}\right\}$ half-plane. Let us take $z \in \mathbb{C}^{+}$, the upper half of the complex plane ($\text{Im } z > 0$). As we have established, the triplets $\Phi^{\pm} \subset \mathcal{H} \subset (\Phi^{\pm})^{\times}$ are equivalent to the triplets $\Delta_{\mp} \subset \mathcal{H}^2_{\pm} \subset \Delta^{\times}_{\mp}$, respectively, in the sense that $\Phi^{\pm}$ and $\Delta_{\mp}$ are isomorphic from both the algebraic and topological points of view and the same happens with their duals. Therefore, $|z^{+}\rangle$ and $|z^{-}\rangle$ are equal if and only if their corresponding elements in $\Delta^{\times}_{\mp}$ are equal.

What are these elements? To find them, let us recall that the wave function associated to $\phi^{\pm}$ is given by $\phi^{\pm}(\omega) = \langle\omega^{\pm}|\phi^{\pm}\rangle \in \Gamma_{\mp}$, with $\omega > 0$, in the energy representation. The relations between $\Gamma_{\mp}$ and $\Delta_{\mp}$ are given by the operators $\theta_{\mp}$ as $\theta_{\mp}^{-1}\phi^{\pm}(\omega) := h^{\pm}(\omega) \in \Delta_{\mp}$. The functions $h^{\pm}(\omega)$ coincide with $\phi^{\pm}(\omega)$ for $\omega \geq 0$. For any other value ω', $h^{\pm}(\omega')$ represents the value of the analytic continuation of $\phi^{\pm}(\omega)$ at ω'. The functionals that we are looking for, $|F^{\pm}\rangle$, are defined as

$$\langle\phi^{\pm}|z^{\pm}\rangle = [\phi^{\pm}(z^*)]^* = \langle h^{\pm}|F^{\pm}\rangle \tag{3.20}$$

We know that $\phi^{-}(\omega) \in \Gamma_{+}$ implies that $[\phi^{+}(\omega)]^* \in \Gamma_{-}$. Equivalently, $h^{-}(\omega) = \theta_{-}^{-1}\phi^{-}(\omega) \in \Delta_{+} \Rightarrow [h^{-}(\omega)]^* \in \Delta_{-}$. If $\text{Im } z > 0$, the Tichmarsh theorem [29] gives

$$[h^{-}(z^*)]^* = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{[h^{-}(\omega)]^*}{\omega - z^*}\, d\omega \tag{3.21}$$

The right-hand side of (3.21) represents the following scalar product on the Hilbert space $L^2(\mathbb{R})$:

$$[h^{-}(z^*)]^* = \left(h^{-}(\omega), \frac{1}{2\pi i}\frac{1}{\omega - z^*}\right) = \langle h^{-}|F^{-}\rangle \tag{3.22}$$

Thus,

$$\left|F^{-}\right\rangle = \frac{1}{2\pi i}\frac{1}{\omega - z^*} \in L^2(\mathbb{R}) \tag{3.23}$$

Analogously, if $\phi^+(\omega) \in \Gamma_- \Rightarrow [h^+(\omega)]^* \in \Delta_+$. Since Im $z^* < 0$, the Tichmarsh theorem reads in this case

$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{[h^+(\omega)]^*}{\omega - z^*}\, d\omega = 0 \tag{3.24}$$

and, therefore, $\left|F^+\right\rangle$ *cannot* be represented by the same vector as $\left|F^-\right\rangle$ in (3.23), since $\langle h^+\left|F^+\right\rangle$ is different from zero in general.

4. THE GAMOW VECTORS AND THEIR BRACKETS

Here we deal with a scattering resonant system [9]. Resonant scattering can be described as follows: A state is prepared in the remote past and evolves freely. As it enters an interaction region governed by a potential V, at least one metastable state is created.[4] At time $t = 0$, this metastable state ceases to be formed and starts to decay. Then, it is represented by the vector $\psi^+ = \psi^+(0)$. For times $t < 0$, $\psi^+(t) = e^{-itH}\psi^+$, evolves under the action of the total Hamiltonian $H = H_0 + V$.

The corresponding asymptotically free vector [9, 13, 31] is given at $t = 0$ by ψ^{in}. For $t < 0$, we have the outgoing state $\psi^-(t)$. Its time evolution is governed again by H. As the outgoing state leaves the interaction region, it can be observed in the far future as the free vector $\psi^{\text{out}}(t)$. At $t = 0$, this free outgoing vector would be $\psi^{\text{out}} = \psi^{\text{out}}(0)$. The relations between these vectors are $\psi^+ = \psi^-$, $\Omega_+\psi^{\text{in}} = \psi^+$, $\Omega_-\psi^{\text{out}} = \psi^-$, and $\psi^{\text{out}} = S\psi^{\text{in}}$.

What we really observe is not ψ^{out}, but rather the projection of ψ^{out} into the region in which the *detector is placed*. The state resulting from this procedure is called φ^{out} [30].

Now, we make our *fundamental Ansatz*: $\psi^{\text{in}} \in \Phi_+$ and $\varphi^{\text{out}} \in \Phi_-$, which is justified by the denseness of Φ_+ and Φ_- in $\mathcal{H}$. Denseness implies that, given an arbitrary vector in $\mathcal{H}$, we always can choose another vector in the dense subspace such that the difference between them is negligible. Our Ansatz implies that $\psi^+ \in \Phi^+$ and $\varphi^- \in \Phi^-$.

The transition amplitude between ψ^{out} and φ^{out} is given by

$$(\varphi^{\text{out}}, S\psi^{\text{in}}) = (\varphi^-, \psi^+) = \int_0^{\infty} S(\omega + i0)\, \langle\varphi^-\left|\omega^-\right\rangle\langle\omega^+\left|\psi^+\right\rangle\, d\omega \tag{4.1}$$

The integral in (4.1) can be decomposed in terms of an integral along

[4] For the sake of simplicity we introduce only one metastable state or pole.

the negative part of the real axis in the second sheet plus a pole term corresponding to the residue of the function under the integral sign on the pole of $S_{\mathrm{II}}(\omega)$ on the lower half-plane (second sheet). This decomposition has already been described.

Now, we introduce the so called "G curve" that encloses both poles z_R and z_R^* as depicted in Fig. 2. Due to the Cauchy theorem, the integral in (4.1) can be written as

$$(\varphi^-, \psi^+) = -2\pi i\{a_0^*\langle\varphi^-|z_R^*\rangle\langle z_R^+|\psi^+\rangle + a_0\langle\varphi^-|z_R^-\rangle\langle z_R^{*+}|\psi^+\rangle\} + \int_G S_{\mathrm{II}}(z)\langle\varphi^-|z^-\rangle\langle z^{*+}|\psi^+\rangle\, dz \tag{4.2}$$

where a_0 is the residue of $S_{\mathrm{II}}(z)$ at z_R. Its complex conjugate a_0^* is the residue of $S_{\mathrm{II}}(z)$ at z_R^* [9, 11]. Without loss of generality, one also may assume that ψ^{in} may be any vector in Φ_+ and that φ^{out} is any vector in Φ_-, which implies that φ^- and ψ^+ can be arbitrarily chosen in Φ^- and Φ^+, respectively. If we omit these two vectors, we can write (4.2), after a redefinition of the functionals that absorbs the constants, as

$$I = |z_R^{*-}\rangle\langle z_R^+| + |z_R^-\rangle\langle z_R^{*+}| + \int_G S_{\mathrm{II}}(z)|z^-\rangle\langle z^{*+}|\, dz \tag{4.3}$$

Since Φ^+ and Φ^- are both contained in $\mathcal{H}_{\mathrm{ac}}$, which is contained in both

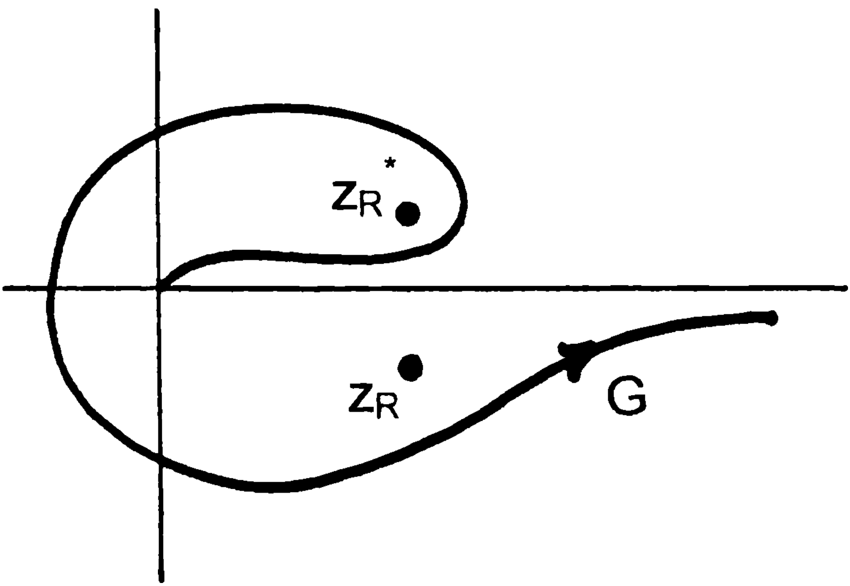

Fig. 2.

$(\Phi^+)^\times$ and $(\Phi^-)^\times$, then $\Phi^+ \subset (\Phi^-)^\times$ and $\Phi^- \subset (\Phi^+)^\times$. Thus, the identity in (4.3) can be interpreted as the canonical mapping I: $\Phi^+ \mapsto (\Phi^-)^\times$, which maps any vector in Φ^+ into the same vector considered as an element of $(\Phi^-)^\times$. The formal product of (4.3) by H either to the right or to the left gives

$$H = z_R^* \big|z_R^{*-}\rangle\langle z_R^+\big| + z_R\big|z_R^-\rangle\langle z_R^{*+}\big| + \int_G zS_{\rm II}(z)\big|z^-\rangle\langle z^{*+}\big|\, dz \tag{4.4}$$

We have already studied the behavior of state vectors of the kind $\big|z^\pm\rangle$ and $\langle z^\pm\big|$. When $z = z_R$ or $z = z_R^*$ the state vectors $\big|z_R^-\rangle$ and $\big|z_R^{*+}\rangle$ have a clear physical meaning.[5] They are the vectors that describe the exponentially decaying and the exponentially growing part of a resonance [9,11]. Formula (4.4) is a generalized spectral decomposition of the total Hamiltonian in terms of projections including these vectors. They are called the decaying and the growing *Gamow vectors*, respectively. The operators in the spectral decompositions of (4.3) and (4.4) are truly (non-self-adjoint) projections, as we shall show in the sequel. In order to do that, we need to define certain brackets. They have a correct meaning as distributional kernels and can be obtained as follows: Let us multiply (4.3):

(i) To the right by $\big|z_R^-\rangle$,

$$\big|z_R^-\rangle = \big|z_R^{*-}\rangle\langle z_R^+\big|z_R^-\rangle + \big|z_R^-\rangle\langle z_R^{*+}\big|z_R^-\rangle + \int_G S_{\rm II}(z)\big|z^-\rangle\langle z^{*+}\big|z_R^-\rangle\, dz \tag{4.5}$$

which says that

$$\boxed{\langle z_R^+\big|z_R^-\rangle = 0}\,; \qquad \langle z_R^{*+}\big|z_R^-\rangle = 1; \qquad \langle z^{*+}\big|z_R^-\rangle = 0 \tag{4.6}$$

(ii) To the right by $\big|z_R^{*-}\rangle$,

$$\big|z_R^{*-}\rangle = \big|z_R^{*-}\rangle\langle z_R^+\big|z_R^{*-}\rangle + \big|z_R^-\rangle\langle z_R^{*+}\big|z_R^{*-}\rangle + \int_G S_{\rm II}(z)\big|z^-\rangle\langle z^{*+}\big|z_R^{*-}\rangle\, dz \tag{4.7}$$

which implies that

$$\langle z_R^+\big|z_R^{*-}\rangle = 1; \qquad \boxed{\langle z_R^{*+}\big|z_R^{*-}\rangle = 0}\,; \qquad \langle z^{*+}\big|z_R^{*-}\rangle = 0 \tag{4.8}$$

(iii) To the left by $\langle z_R^{*+}\big|$, we obtain

$$\langle z_R^{*+}\big|z_R^{*-}\rangle = 1; \qquad \boxed{\langle z_R^{*+}\big|z_R^-\rangle = 0}\,; \qquad \langle z_R^{*+}\big|z^-\rangle = 0 \tag{4.9}$$

(iv) To the left by $\langle z_R^{*+}\big|$, we get

[5] This is not the case for $\big|z_R^{*-}\rangle$ and $\big|z_R^+\rangle$.

$$\boxed{\langle z_R^{*+} | z_R^{*-} \rangle = 0}\,; \qquad \langle z_R^{*+} | z_R^- \rangle = 1; \qquad \langle z_R^{*+} | z^- \rangle = 0 \tag{4.10}$$

Observe that (4.9) and (4.10) provide us with the same formulas as (4.6) and (4.8). Hence, multiplication by the right does not give us any new result.

To obtain the desired new results, one should consider the scalar product given by

$$\begin{aligned}(\psi^+, \varphi^-) &= \int_0^\infty S(\omega - i0)\langle \psi^+ | \omega^+ \rangle \langle \omega^- | \varphi^- \rangle \, d\omega \\ &= 2\pi i \{ a_0 \langle \psi^+ | z_R^{*+} \rangle \langle z_R^- | \varphi^- \rangle + a_0^* \langle \psi^+ | z_R^+ \rangle \langle z_R^{*+} | \varphi^- \rangle \} \\ &\quad + \int_{\tilde{G}} S_{\text{II}}(z) \langle \psi^+ | z^{*+} \rangle \langle z^- | \varphi^- \rangle \, dz \end{aligned} \tag{4.11}$$

which is the complex conjugate of (4.2). The curve $\tilde{G}$ is depicted in Fig. 3. It lies in the second sheet. The first integral in (4.11) is extended over the lower rim of the cut.

As we deal with the second sheet only, we can forget about the Riemann surface and work on a complex plane. The function given by $S_{\text{II}}(z)$ has, in addition to the resonant poles, a branch cut along the positive part of the real axis. The boundary values of $S_{\text{II}}(z)$ on $\mathbb{R}^+$ from above coincide with

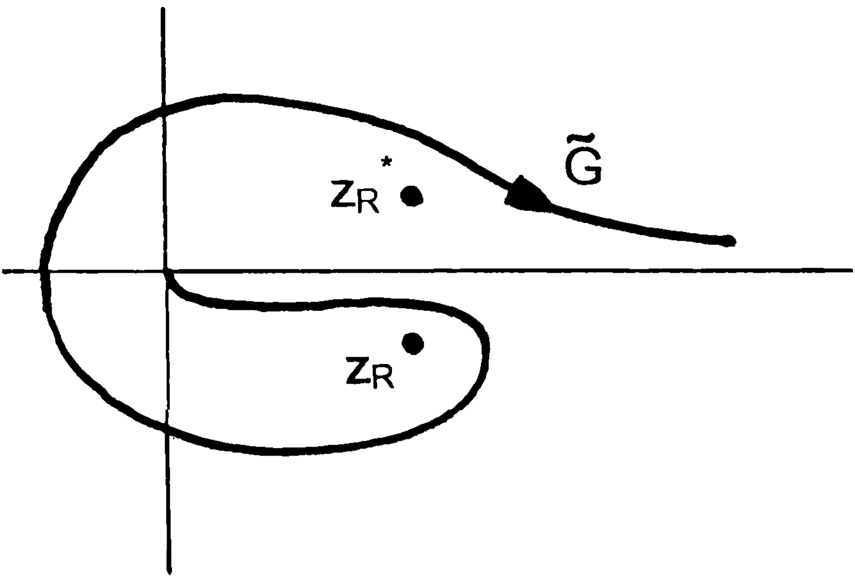

Fig. 3.

$S(\omega - i0)$ and those from below with $S(\omega + i0)$, a situation opposite to that in the first sheet.

Equation (4.11) can be written formally, after reabsorbing constants, as

$$I = |z_R^{*+}\rangle\langle z_R^-| + |z_R^+\rangle\langle z_R^{*-}| + \int_{\tilde{G}} S_{II}(z)|z^{*+}\rangle\langle z^-|\, dz \tag{4.12}$$

I is the canonical mapping $I: \Phi^- \mapsto (\Phi^+)^\times$ defined as in the previous case. Note that (4.12) is the adjoint of (4.3) in some sense. If, as before, we apply this expression to the Gamow vectors, we obtain new brackets. Thus, if we multiply:

(i) To the right by $|z_R^{*+}\rangle$, we get

$$|z_R^{*+}\rangle = |z_R^{*+}\rangle\langle z_R^-|z_R^{*+}\rangle + |z_R^+\rangle\langle z_R^{*-}|z_R^{*+}\rangle + \int_{\tilde{G}} S_{II}(z)|z^{*+}\rangle\langle z^-|z_R^{*+}\rangle\, dz \tag{4.13}$$

which gives

$$\langle z_R^-|z_R^{*+}\rangle = 1; \qquad \boxed{\langle z_R^{*-}|z_R^{*+}\rangle = 0}; \qquad \langle z^-|z_R^{*+}\rangle = 0 \tag{4.14}$$

(ii) Formal multiplication to the right by $|z_R^+\rangle$ gives analogously

$$\boxed{\langle z_R^-|z_R^+\rangle = 0}; \qquad \langle z_R^{*-}|z_R^+\rangle = 1; \qquad \langle z^-|z_R^+\rangle = 0 \tag{4.15}$$

(iii) Multiplication to the left by $\langle z_R^-|$ and $\langle z_R^{*-}|$ leads again to (4.14) and (4.15).

After (4.12), the adjoint of (4.4) is given by

$$H = z_R^*|z_R^{*+}\rangle\langle z_R^-| + z_R|z_R^+\rangle\langle z_R^{*-}| + \int_{\tilde{G}} zS_{II}(z)|z^{*+}\rangle\langle z^-|\, dz \tag{4.16}$$

As mentioned earlier, the brackets that we have found so far have a rigorous meaning in terms of distributions. They allow us to show that the spectral decompositions of H given by (4.4) and (4.16) are written in terms of projections. The property that defines a projection is idempotency: $\Pi^2 = \Pi$.

Thus, take, for instance, $\Pi = |z_R^{*-}\rangle\langle z_R^+|$. Then, using the first identity in (4.8), we have

$$\Pi^2 = |z_R^{*-}\rangle\langle z_R^+|z_R^{*-}\rangle\langle z_R^+| = \Pi \tag{4.17}$$

The idempotency of $|z_R^-\rangle\langle z_R^{*+}|$, $|z_R^{*+}\rangle\langle z_R^-|$, and $|z_R^+\rangle\langle z_R^{*-}|$ is proven analogously. These projections are, in principle, good candidates for being density states for resonance states or *Gamow* densities. We discuss briefly this point later.

The integral term in (4.3) and (4.12) has similar properties. Let us study briefly this term in (4.3). An analogous study can be done for (4.12). Let us define

$$|f_z\rangle := \sqrt{S_{II}(z)}|z^-\rangle \qquad \text{and} \qquad |\tilde{f}_z\rangle := \sqrt{S_{II}(z)}z^{*+}\rangle \tag{4.18}$$

Then, (4.3) can be written as

$$I = |z_R^{*-}\rangle\langle z_R^+| + |z_R^-\rangle\langle z_R^{*+}| + \int_G |f_z\rangle\langle\tilde{f}_z|\, dz \tag{4.19}$$

Multiplying to the right (4.19) by $|f_\omega\rangle$ where $\omega \in G$, we obtain

$$|f_\omega\rangle = |z_R^{*-}\rangle\langle z_R^+|f_\omega\rangle + |z_R^-\rangle\langle z_R^{*-}|f_\omega\rangle + \int_G |f_z\rangle\langle\tilde{f}_z|f_\omega\rangle\, dz \tag{4.20}$$

which means that

$$\langle z_R^+|f_\omega\rangle = 0; \qquad \langle z_R^{*-}|f_\omega\rangle = 0; \qquad \langle\tilde{f}_z|f_\omega\rangle = \delta_G(z-\omega) \tag{4.21}$$

The meaning of the delta in (4.21) is obvious. These formulas imply that

$$\langle z_R^+|z^-\rangle = \langle z_R^{*-}|z^-\rangle = 0 \tag{4.22}$$

If we multiply (4.19) to the left by $\langle\tilde{f}_\omega|$, we obtain

$$\langle z^{*+}|z_R^{*-}\rangle = \langle z^{*+}|z_R^-\rangle = 0 \tag{4.23}$$

Similar manipulations with (4.12) give

$$\langle z_R^-|z^{*+}\rangle = \langle z_R^{*-}|z^{*+}\rangle = 0 \tag{4.24}$$

$$\langle z^-|z_R^{*+}\rangle = \langle z^-|z_R^+\rangle = 0 \tag{4.25}$$

To define new brackets, one should perform more formal operations of the same type. Let us multiply (4.4) to the right by $|z_R^+\rangle$ and to the left by $\langle z_R^+|$. We obtain

$$z_R\langle z_R^+|z_R^+\rangle = z_R^*\langle z_R^+|z_R^{*-}\rangle\langle z_R^+|z_R^+\rangle + z_R\langle z_R^+|z_R^-\rangle\langle z_R^{*+}|z_R^+\rangle$$
$$+ \int_G zS_{II}(z)\langle z_R^+|z^-\rangle\langle z^{*+}|z_R^+\rangle\, dz \tag{4.26}$$

which, after (4.6), (4.8), and (4.9) gives

$$\boxed{\langle z_R^+|z_R^+\rangle = 0} \tag{4.27}$$

because then the right-hand side of (4.26) is equal to $z_R^*\langle z_R^+|z_R^+\rangle$ and this gives directly $\langle z_R^+|z_R^+\rangle = 0$. With manipulations of the same type, we obtain

$$\boxed{\langle z_R^-|z_R^-\rangle = 0; \qquad \langle z_R^{*-}|z_R^{*-}\rangle = 0; \qquad \langle z_R^{*+}|z_R^{*+}\rangle = 0} \tag{4.28}$$

Formulas (4.27) and (4.28) are compatible with an idea, original to Nakanishi [32] and mentioned by other authors [25, 33, 34, 12], according to which Gamow vectors *should have a zero norm*. In Hilbert space, only the zero vector can have zero-norm. Therefore, those authors are not talking about a norm in the usual sense. Roughly speaking, the argument goes as follows:

$$0 = \langle z_R^- | (H - H) | z_R^- \rangle = (z_R^* - z_R)\langle z_R^- | z_R^- \rangle \tag{4.29}$$

Since z_R has nonzero imaginary part, this implies that $\langle z_R^- | z_R^- \rangle = 0$ from a formal point of view. This argument assumes implicitly that $|z^-\rangle$ belongs to the domain of the operator H, which is false. In any case, Nakanishi's demonstration is incomplete, since no definition was given at that time for the new eigenstate. We must insist that these brackets are not scalar products in the usual sense. The rigorous nature of these brackets is the subject of our investigations and may be related to a generalized concept of trace on rigged Liouville spaces.

For the sake of completeness we will formally compute the missing products $\langle z_R^{*+} | z_R^+ \rangle$, $\langle z_R^{*-} | z_R^- \rangle$, $\langle z_R^+ | z_R^{*+} \rangle$, $\langle z_R^- | z_R^{*-} \rangle$. [Note that $\langle z_R^{*+} | z_R^+ \rangle = \langle z_R^+ | z_R^{*+} \rangle^*$ and $\langle z_R^{*-} | z_R^- \rangle = \langle z_R^- | z_R^{*-} \rangle^*$.]

In (4.14) we obtained that $\langle z_R^- | z_R^{*+} \rangle = 1$. Now, if we insert the identity I as in (4.3) and use again the identities (4.14), we have that

$$1 = \langle z_R^- | z_R^{*-} \rangle = \langle z_R^- | z_R^{*-} \rangle \langle z_R^+ | z_R^{*+} \rangle \tag{4.30}$$

The symmetry between decaying and growing processes implies that

$$\langle z_R^- | z_R^{*+} \rangle = \pm \langle z_R^+ | z_R^{*+} \rangle \tag{4.31}$$

Then, (4.30) and (4.31) imply

$$\langle z_R^- | z_R^{*-} \rangle = \pm 1 \text{ or } \pm i \tag{4.32}$$

and therefore

$$\left| \langle z_R^- | z_R^{*-} \rangle \right| = 1 \tag{4.33}$$

The formalism does not fix the phases, which we conventionally take to be equal to zero. Thus, $\langle z_R^- | z_R^{*-} \rangle = 1$. Analogously, $\langle z_R^+ | z_R^{*+} \rangle = 1$

Final Remark. Consider formula (4.1) in which we omit φ^- and ψ^+:

$$I = \int_0^\infty S(\omega + i0) |\omega^-\rangle\langle\omega^+| \, d\omega \tag{4.34}$$

and also [see (2.8)]

$$I = \int_0^\infty \big|\omega^+\rangle\langle\omega^+\big|\, d\omega \tag{4.35}$$

These identities are not equivalent. The first one is the identity I: $\Phi^+ \mapsto (\Phi^-)^\times$ and the second one is I: $\Phi^+ \mapsto (\Phi^+)^\times$. Due to this difference, one cannot, in principle, identify $|\omega_+\rangle$ with $S(\omega + i0)|\omega_-\rangle$. However, since the operator S is untary, the identity $|\omega_+\rangle = S(\omega + i0)|\omega_-\rangle$ could have an interpretation within the following context: Since $f_+ \in \Phi_+$ represents an incoming free vector, one should find an outgoing free vector f_- such that $f_- = Sf_+$. Then, one may define $S|\omega_+\rangle$ as

$$\langle f_-|S|\omega_-\rangle = \langle Sf_+|\omega_+\rangle, \qquad \forall f_+ \in \Phi_+ \tag{4.36}$$

In order to define correctly the arguments in (4.36), we must have that $f_- = Sf_+ \in \Phi_-$, which again implies that $\Gamma_- = S(\omega + i0)\Gamma_+$. This cannot be true in general, and hence $|\omega_+\rangle = S|\omega_-\rangle$ is false in general. $S\Phi_+$ is a space which is different from Φ_-. This does not represent a difficulty for our formalism due to the *denseness* of both spaces as we have argued elsewhere. The functional $S(\omega + i0)|\omega_-\rangle$ lies in the space $(S\Phi_+)^\times$.

However, the formula $|\omega^-\rangle = S(\omega + i0)|\omega^+\rangle$ has a well-defined meaning on $\Phi^+ \cap \Phi^-$ [34].

5. TIME EVOLUTION OF THE GAMOW STATES. IRREVERSIBILITY

The formulas giving the time evolution of the Gamow states are a direct consequence of some formulas obtained in Section 3. For the sake of completeness, let us write here the relevant equations concerning the Gamow vectors. The first two among these equations can be obtained as a particular case of (3.1) just by replacing z and z^* by z_R and z_R^*, respectively. We have

$$H|z_R^{*\pm}\rangle = z_R^*|z_R^{*\pm}\rangle; \qquad H|z_R^\pm\rangle = z_R|z_R^\pm\rangle \tag{5.1}$$

Analogously, as a particular case of (3.14), we have

$$\langle z_R^\pm|H = z_R^*\langle z_R^\pm|; \qquad \langle z_R^{*\pm}|H = z_R\langle z_R^{*\pm}| \tag{5.2}$$

We recall that the points z_R and z_R^* are the pair of poles on the second sheet of the S-matrix in the energy representation which define the resonance. Thus, $z_R = E_R - (i\Gamma_R)/2$, $z_R^* = E_R + (i\Gamma_R)/2$, where E_R represents the resonant energy and Γ_R the width, so that $2/\Gamma_R$ is the mean life of the resonance state.

The time evolution of the Gamow vectors is obtained from formulas (3.15)–(3.18). For $t > 0$, we have

$$\left|z_R^-(t)\right\rangle := e^{-itH}\left|z_R^-\right\rangle = e^{-iz_Rt}\left|z_R^-\right\rangle = e^{-iE_Rt}e^{-\Gamma_Rt/2}\left|z_R^-\right\rangle, \qquad t > 0 \qquad (5.3)$$

As we can see from (5.3), $\left|z_R^-(t)\right\rangle$ decays exponentially with time. Thus, it is the state vector for the exponentially decaying state, which was created up to $t = 0$. It is called the decaying Gamow vector for the resonance at z_R.

Analogously, for $t > 0$, we have

$$\left|z_R^{*-}(t)\right\rangle := e^{-itH}\left|z_R^{*-}(t)\right\rangle = e^{-itE_R}\, e^{\Gamma_Rt/2}\left|z_R^{*-}\right\rangle, \qquad t > 0 \qquad (5.4)$$

This vector blows up as $t \mapsto \infty$. We call it the anomalous growing Gamow vector. The word anomalous stresses the fact that this vector grows exponentially for positive values of time.

It is also very interesting to point out that the time evolution of the vectors $\left|z_R^-\right\rangle$ and $\left|z_R^{*-}\right\rangle$ is not defined for $t < 0$. The reason is clear: e^{-itH} cannot be extended to the dual space $(\Phi^-)^\times$ [10].

For $t < 0$, we have

$$\left|z_R^{*+}(t)\right\rangle := e^{-itH}\left|z_R^{*+}(t)\right\rangle = e^{-itE_R}\, e^{\Gamma_Rt/2}\left|z_R^{*+}\right\rangle, \qquad t < 0 \qquad (5.5)$$

Thus, $\left|z_R^{*+}(t)\right\rangle$ grows exponentially with time up to $t = 0$. It is the exponentially growing Gamow vector. It represents the state of a forming resonance, which ceases to be formed at $t = 0$ [8]. It is interesting to note that the time behavior of the anomalous growing Gamow vector $\left|z_R^{*-}(t)\right\rangle$ is the continuation to $t > 0$ of the time behavior in the region $t < 0$ of $\left|z_R^{*+}(t)\right\rangle$. In this sense, the anomalous growing Gamow vector is the unphysical continuation of the physical growing Gamow vector. The anomalous growing Gamow vector is therefore a mathematical object that appears in the spectral decompostions, although it does not occur in nature. We shall find some use for this vector in Section 7.

Analogously, for $t < 0$, we have

$$\left|z_R^+(t)\right\rangle := e^{-itH}\left|z_R^+(t)\right\rangle = e^{-iE_Rt}\, e^{-\Gamma_Rt/2}\left|z_R^+\right\rangle, \qquad t < 0 \qquad (5.6)$$

This vector blows up as $t \mapsto -\infty$. It is called the anomalous decaying Gamow vector. It decays as time grows up to $t = 0$. As for the case of the anomalous growing Gamow vector and in the same sense, the anomalous decaying Gamow vector is the unphysical continuation to $t < 0$ of the physical decaying Gamow vector $\left|z_R^-(t)\right\rangle$ defined for $t > 0$ only. It also will be used below.

Time evolution for the Gamow vectors $\left|z_R^{*+}(t)\right\rangle$ and $\left|z_R^+(t)\right\rangle$ is well defined for $t < 0$ only, since e^{-itH} cannot be extended to $(\Phi^+)^\times$ for $t > 0$.

Resonance states appear with resonances characterized as pairs of poles on the second sheet of the S-matrix in the energy representation. Therefore, they are physically relevant if and only if resonances are present. Only when resonances exist do the dual spaces $(\Phi^-)^\times$ and $(\Phi^+)^\times$ acquire full meaning and they are necessary. In fact, the Gamow vectors $\left|z_R^{*+}\right\rangle$ and $\left|z_R^-\right\rangle$ representing resonance states cannot exist in a Hilbert space [10]. We have seen that on these dual spaces the evolution group e^{-itH} splits into two semigroups, one corresponding to the past $t < 0$ and the other to the future $t > 0$. *Since the splitting is a mathematical formulation for time asymmetry and irreversible evolution*, we have extended a formulation for irreversibility in quantum systems with resonances [25, 34]. The connection between this irreversibility and entropy increasing will be sketched later.

The choice of the spaces Φ^- and Φ^+ is essential in this formalism. In this choice, Hardy functions play a fundamental role. In view of this situation, the following question arises: why Hardy functions? The answer lies in the very nature of a scattering phenomenon and was formulated by Bohm, and Antoniou [13] and Bohm *et al.* [31]. The main purpose of these authors was to define a quantum arrow of time for scattering experiments. In this kind of experiment they are essentially three kinds of processes: preparation of an incoming free state, scattering with a center of forces, and registration of the projection of the outgoing free state into the region occupied by the registration apparatus. Then, one chooses the time $t = 0$ as the time at which all preparations have been completed and after which the registration begins (see Section 8). In particular, this means that the incoming wave function must be zero for $t > 0$, which implies that for $t > 0$ [13]

$$0 = \int_{-\infty}^{\infty} d\omega \, \langle \omega_+ | \varphi^{\text{in}}(t) \rangle = \int_{-\infty}^{\infty} d\omega \, e^{-it\omega} \langle \omega^+ | \varphi^+ \rangle, \qquad t > 0 \qquad (5.7)$$

where ω denotes the enrgy. Then, the Paley–Wienner theorem [51] shows that (5.7) implies that $\langle \omega^+ | \varphi^+ \rangle$ (with $-\infty < \omega < \infty$) is a Hardy function.

Since no preparations take place *after* $t = 0$, the outgoing state vector $\psi^{\text{out}}(t)$ is zero for $t < 0$. Using the same arguments, one sees that $\langle \omega^- | \psi^- \rangle$ must be a Hardy function.

We see that the above arguments are closely connected with causality. Therefore, causality arguments determine the use of Hardy functions. This is not new, since the use of $\mathcal{H}^p$ functions in descriptions of scattering processes has been motivated through causality conditions [13, 31]. Note that they also appear in the Lax–Phillips theory of scattering [36].

We can also find the time evolution for the left eigenvectors in an analogous manner as we found the time evolution for the right eigenvectors. This time evolution is given by means of the following formulas:

For $t > 0$

$$\langle z_R^-(t) | := \langle z_R^- | e^{itH} = e^{iz_R^* t} \langle z_R^- | = e^{iE_R t} \, e^{\Gamma_R t/2} \langle z_R^- | \qquad (5.8)$$

$$\langle z_R^{*-}(t) | := \langle z_R^{*-} | e^{itH} = e^{iE_R t} \, e^{-\Gamma_R t/2} \langle z_R^{*-} | \qquad (5.9)$$

For $t < 0$

$$\langle z_R^{*+}(t)| := \langle z_R^{*+}| e^{itH} = e^{itE_R} e^{-\Gamma_R t/2} \langle z_R^{*+}| \tag{5.10}$$

$$\langle z_R^{+}(t)| := \langle z_R^{+}| e^{itH} = e^{iE_R t} e^{\Gamma_R t/2} \langle z_R^{+}| \tag{5.11}$$

To finish this section, let us make some comments on the formulation of decaying states in terms of density matrices. Consider a radiative sample. It is an unstable system tending toward a state of equilibrium. In the state of equilibrium all the atoms in the sample have already decayed. From a phenomenological point of view, the time evolution of the state of the sample should have the following form:

$$\rho(t) = \rho_* + \rho_1 e^{-\Gamma t} \tag{5.12}$$

where ρ_* denotes the state of equilibrium of the sample and the second term in the right-hand side of (5.12) is a correction to the equilibrium that vanishes exponentially with a mean life Γ^{-1}. The equilibrium state ρ_* as well as $\rho(t)$ must be normalized. However, the term ρ_1 presents several difficulties. First of all, to date nobody has consistently defined a density operator for a decaying state like ρ_1. Therefore, we do not know how to define a trace for ρ_1. But if $\mathrm{tr}\rho(t) = 1 = \mathrm{tr}\rho_*$, one must have

$$\mathrm{tr}\rho_1 = 0 \tag{5.13}$$

Which object would be a good candidate for ρ_1? Several possibilities have been proposed for defining a density operator of a decaying state, none of them totally free of inconsistencies [36, 37]. However, an excellent candidate for a model with one resonance only is

$$\rho_1 = |z_R^-\rangle\langle z_R^-| \tag{5.14}$$

for which the time evolution can be easily defined for $t > 0$ as

$$\rho_1(t) = e^{-itH}|z_R^-\rangle\langle z_R^-|e^{itH} \tag{5.15}$$

This definition coincides with the standard rule for the quantum evolution of the states. Thus, using (5.3) and (5.8), we obtain

$$\rho(t) = \rho_* + e^{-\Gamma_R t}\rho_1 \tag{5.16}$$

Then, we can define the trace of ρ_1 with the aid of (4.28) so that

$$\mathrm{tr}\rho_1 := \langle z_R^-|z_R^-\rangle = 0 \tag{5.17}$$

as expected.

6. TIME REVERSAL AND IRREVERSIBILITY

The Wigner time inversion operator K has the following properties [37]:

1. Antilinearity.
2. If Q and P are the position and momentum operators, respectively, one has

$$KQK = Q; \qquad KPK = -P \tag{6.1}$$

3. If $\psi(x)$, a function of the position, is the wave function of a given quantum state, then $K\psi(x) = \psi^*(x)$, where the star denotes complex conjugation. As a consequence, if $\varphi(\omega)$ represents the same state in the energy representation, we also have $K\varphi(\omega) = \varphi^*(\omega)$.
4. $K^2 = I$, the identity, for spin-zero states.

Property 3 implies that K is a one-to-one mapping from $\Gamma_\pm$ onto $\Gamma_\mp$. One can show that it is also continuous. Then, we can define time-reversal operators $T_\pm$ from $\Phi^\pm$ onto $\Phi^\mp$ as follows: Consider the operator U defined in Section 2 and the operators $\Omega_\pm$. Then, we know that the mappings $V_\pm = U\Omega_\pm^\dagger$ give an equivalence between topological vector spaces: $V_\pm$: $\Gamma_\mp \mapsto \Phi^\pm$. Then, one defines $T_\pm := V_\mp K V_\pm^\dagger$, where $V_\pm = \Omega_\pm U^{-1}$. Obviously, this definition makes the following diagram commutative:

$$\begin{array}{ccc} \Gamma_\mp & \overset{K}{\to} & \Gamma_\pm \\ V_\pm\downarrow & & \downarrow V_\mp \\ \Phi^\pm & \overset{T_\pm}{\to} & \Phi^\mp \end{array} \tag{6.2}$$

Thus, each of the $T_\pm$ is a continuous one-to-one mapping from $\Phi^\pm$ onto $\Phi^\mp$. In addition, they have the following properties:

(i) By construction, they are antilinear. Since $\Phi^\pm$ are dense in $\mathcal{H}_{ac}$ and their topologies are finer than the topology they have as subspaces of $\mathcal{H}_{ac}$, $T_\pm$ can be extended to antiunitary operators on $\mathcal{H}_{ac}$.

(ii) Their adjoints are given by

$$T_\pm^\dagger = V_\pm K^\dagger V_\mp^\dagger = V_\pm K V_\mp^\dagger = T_\mp \tag{6.3}$$

(iii) The operators T_+ and T_- are the inverses of one another:

$$T_+T_- = V_- \; KV_+^\dagger V_+ \; KV_-^\dagger = I = T_-T_+ \tag{6.4}$$

Now, let us consider an arbitrary $\phi^\pm \in \Phi^\pm$. Its wave function in the energy representation is given by

$$\phi^\pm(\omega) = \langle\omega^\pm|\phi^\pm\rangle = [V_\pm^\dagger\phi^\pm](\omega) \in \Gamma_\mp \tag{6.5}$$

The definition of $T_\pm$ and (6.5) give for any $\omega \in \mathbb{R}^+$:

$$[T_{\pm}\phi^{\pm}](\omega) = [V_{\mp}^{\dagger}T_{\pm}\phi^{\pm}](\omega) = K(V_{\mp}^{\dagger}\phi^{\pm})(\omega) = K\phi^{\pm}(\omega) = [\phi^{\pm}(\omega)]^* \quad (6.6)$$

Hence,

$$\langle\omega^{\mp}|T_{\pm}\phi^{\pm}\rangle = (T_{\pm}\phi^{\pm})(\omega) = [\phi^{\pm}(\omega)]^* = \langle\phi^{\pm}|\omega^{\pm}\rangle \quad (6.7)$$

If $\Phi \subset \mathcal{H} \subset \Phi^{\times}$ is a RHS and T is a continuous antilinear operator on Φ fulfilling the properties that linear operators have to do in order to be extended to the dual (see Appendix A), it can still be extended to a continuous antilinear operator on $\Phi^{\times}$ by means of the following formula: If $F \in \Phi^{\times}$ and $\phi \in \Phi$ are arbitrary,

$$\langle T^{\dagger}\phi|F\rangle = \langle TF|\phi\rangle \quad (6.8)$$

(6.8) finds its motivation in the definition of the adjoint of an antilinear operator [38].

Thus, (6.8) gives

$$\langle\omega^{\mp}|T_{\pm}^{\dagger}\phi^{\pm}\rangle = \langle\phi^{\pm}|(T_{\mp}|\omega^{\mp}\rangle) = \langle\phi^{\pm}|\omega^{\pm}\rangle \quad (6.9)$$

for any $\phi^{\pm} \in \Phi^{\pm}$, which yields

$$T_{\pm}|\omega^{\pm}\rangle = |\omega^{\mp}\rangle \quad (6.10)$$

Now, let us pick any complex $z \in \mathbb{C}$. According to (6.7) and (2.15), one has for the analytic continuation of the wave function of $T_{\pm}\phi^{\pm}$ at z

$$\langle\phi^{\pm}|z^{*\pm}\rangle = [\phi^{\pm}(z)]^* = (T_{\pm}\phi^{\pm})(z^*) = \langle z^{\mp}|T_{\pm}\phi^{\pm}\rangle \quad (6.11)$$

Thus,

$$\langle z^{\mp}|T_{\mp}^{\dagger}\phi^{\pm}\rangle = \langle\phi^{\pm}|(T_{\mp}|z^{\mp}\rangle) = \langle\phi^{\pm}|z^{*\pm}\rangle \quad (6.12)$$

which yields

$$T_{\pm}|z^{\pm}\rangle = |z^{*\mp}\rangle; \qquad \forall z \in \mathbb{C} \quad (6.13)$$

In particular, (6.13) reads for the Gamow vectors

$$T_{-}|z_R^{-}\rangle = |z_R^{*+}\rangle \qquad \text{and} \qquad T_{+}|z_R^{*+}\rangle = |z_R^{-}\rangle \quad (6.14)$$

In summary, time-reversal operations transform the growing part of a resonant scattering into the decaying part and vice versa. Growing Gamow vectors are mapped into decaying Gamow vectors and conversely. *The operator K which acts on the space of wave functions has the same behavior.* If we forget about analytic continuations to the first sheet, which play no role in the theory of resonances, we see the following: from the point of view of the Riemann surface, the elements of Γ_{-} are boundary values of Hardy functions on the lower half-plane of the second sheet and, hence, defined on the upper rim of the cut. On the other hand, a similar argument shows that the functions in Γ_+ are defined on the lower rim. Then, K maps functions on the upper rim into functions in the lower rim and vice versa. This idea is compatible with the fact that time inversion transforms the linear momentum p into $-p$ and consequently interchanges the rims between themselves. Also, $KS(E + i0)K = S(E - i0)$, the values of the S matrix on the upper and lower rims are therefore interchanged. This gives us a coherent picture. To complete it, let us consider the semigroup e^{-itH} with $t > 0$. One has

$$T_+ \, e^{itH} T_- = V_- K V_+^\dagger \, e^{itH} V_+ K V_-^\dagger = V_- K \, e^{itE} \, K V^\dagger{}_- \\ = V_- \, e^{-itE} \, V_-^\dagger = e^{-itH} \, (= e^{itH} \text{ with } t < 0) \qquad (6.15)$$

In (6.15), the second and fourth identities are obvious due to (6.2). The third one is a consequence of the antilinearity of K. After (6.15), we see that time-reversal operators interchange the dynamical semigroups corresponding to the growing and the decaying processes.

Final Remark. This point of view however, is not universally accepted. In particular, Bohm [56] considers that it does not represent a complete picture of the time reversal for an irreversible phenomenon. Take, for instance, the following situation: In a scattering experiment the prepared incoming state is the combination of two uncorrelated (prepared with two independent apparatuses) plane waves. After scattering with each other, the outgoing state consists of coherent spherical waves. The time reversal of this process is another process consisting of preparing a system with two highly correlated spherical waves (with a fixed relative phase) such that, after collision with each other, two uncorrelated plane waves result. It is highly improbable if not impossible to prepare such a state [36]. Therefore, in a scattering experiment there exist states for which we cannot prepare their time-reversed states.

According to Bohm's point of view, the space Φ_+ (or $\Phi_+^\times$) is the space of those states that can be prepared and the space Φ_- (or $\Phi_-^\times$) represents the possible observed properties. This picture has been elaborated from previous ideas of Ludwig on the foundations of quantum mechanics [37] and they lead to the construction of a quantum arrow of time [31]. From the considerations presented in the previous paragraph, Bohm concludes that Φ_- cannot be the time reversal of Φ_+.

To solve this problem, Bohm recalls that K is not the only candidate for a time-reversal operator in quantum mechanics. In fact, Wigner has shown that there exist three other possibilities to define time inversion. They are related with parity and require a doubling of the space of the states [38, 39].

We do not want to discuss here the effects of this doubling of spaces and how it solves this inconvenience. This discussion is very interesting and, in our opinion, provides some insight into the problem of time reversal and irrversibility. However, it contains a controversial point. In fact, we should recall that that Φ_- is not $S\Phi_+$. As a matter of fact, Bohm's argument shows that Φ_+ is not the time reversal of $S\Phi_+$, which, as we wish to insist, is not Φ_-. The fact that Φ_+ and Φ_- as well as Φ^+ and Φ^- are time reversals of each other is a mathematical result that cannot be denied with arguments of a purely physical character.

7. LYAPUNOV VARIABLES

The formalism we have developed so far permits us to introduce Lyapunov variables, i.e., variables that grow or decrease monotonically in time. We give three examples

For obvious reasons, we call the first of our Lyapunov variables the "survival probability." In order to define it, let us go back to (4.1). We can deform the contour of integration to Γ (see Fig. 3), in order to obtain [9, 10]

$$(\varphi^-, \psi^+) = -2\pi i a_0 \langle \varphi^- | z_R^- \rangle \langle z_R^{*+} | \psi^+ \rangle + \int_\Gamma S_{II}(z) \langle \varphi^- | z^- \rangle \langle z^{*+} | \psi^+ \rangle \, dz \tag{7.1}$$

If we omit the arbitrary $\varphi^- \in \Phi^-$, we can write ψ^+ as an element of $(\Phi^-)^\times$, using the following formula:

$$|\psi^+\rangle = |z_R^-\rangle + \int_\Gamma \alpha(z) |z^+\rangle \, dz \tag{7.2}$$

in which we have reabsorbed constants. Here, the meaning of $\alpha(z)$ is obvious after (7.1). Now, we define at time $t = 0$ the amplitude of correlation between $|z_R^{*+}\rangle$ and ψ^+ as

$$\langle z_R^{*+} | \psi^+ \rangle = \langle z_R^{*+} | z_R^- \rangle + \int_\Gamma \alpha(z) \langle z_R^{*+} | z^- \rangle \, dz \tag{7.3}$$

Formula (4.10) gives $\langle z_R^{*+} | z^- \rangle = 0$ (we should always assume that $\langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle^*$). Also, (4.10) yields $\langle z_R^{*+} | z_R^- \rangle = 1$. Thus, at $t = 0$, the correlation between ψ^+ and $|z_R^{*+}\rangle$ is $|\langle z_R^{*+} | z_R^- \rangle|^2 = 1$, which gives 1 because of the chosen normalization. At any time $t > 0$, the amplitude should be defined as

$$\langle z_R^{*+} | e^{-itH} | \psi^+ \rangle = e^{-itz_R} \langle z_R^{*+} | \psi^+ \rangle = e^{-itz_R} \langle z_R^{*-} | z_R^- \rangle \tag{7.4}$$

Note that ψ^+ is treated as an element of $(\Phi^-)^\times$, so that the action of e^{-itH} on ψ^+ makes sense. The first identity in (7.4) is a consequence of (5.12). Hence, the survival probabibily should be defined as

$$C(t) = |\langle z_R^{*+} | e^{-itH} | \psi^+ \rangle|^2 = e^{-\Gamma t} |\langle z_R^{*+} | z_R^- \rangle|^2 \tag{7.5}$$

As is well known, this exponential behavior of the survival probability does not occur in ordinary quantum mechanics [40], where one has deviations of the exponential law at short times (Zeno effect) as well as large times (Khalfin effect).The function $C(t)$ is our first Lyapunov variable.

In order to construct our second Lyapunov variable, let us apply the evolution semigroup ($t > 0$) to ψ^+ [$= \psi^+(0)$]:

$$\left|\psi^+(t)\right\rangle = e^{-iz_R t}\left|z_R^-\right\rangle + \int_\Gamma \alpha(z) e^{-izt}\left|z^-\right\rangle dz \tag{7.6}$$

Let us consider the following operator:

$$M = \left|z_R^+\right\rangle\left\langle z_R^+\right| + \int_\Gamma \left|z^+\right\rangle\left\langle z^+\right| dz \tag{7.7}$$

This is an entropy operator in the sense of Misra and Prigogine [41]. Now, let us define the following function:

$$Y = -\left\langle\psi^+(t)\right|M\left|\psi^+(t)\right\rangle \tag{7.8}$$

One finds that

$$Y = -\int_\Gamma \left|\alpha(z)\right|^2 dz - e^{-\Gamma t}\left|\left\langle z_R^+\middle|z_R^-\right\rangle\right|^2 \tag{7.9}$$

After ref. 41, the Liapunov function Y obeys some properties of the entropy function and therefore it can be interpreted as a form of entropy [33].

A more ambitious goal would be to find contractive evolutions, namely evolutions with ever-decreasing norm. Then, we can consider this norm as a Lyapunov variable. These evolutions can be found in Hilbert space $\mathcal{H}$ if we use our generalized vectors. To do this let us define a family of vectors in space $(\Phi^\pm)^\times$,

$$\left|z_C^\pm\right\rangle = \alpha\left|z_R^{*\pm}\right\rangle + \beta\left|z_R^\pm\right\rangle \tag{7.10}$$

functions of variables α and β. Here, we are using both regular and anomalous Gamow vectors.

The norm of these vectors reads

$$\left\langle z_C^\pm\middle|z_C^\pm\right\rangle = \alpha^*\beta\left\langle z_R^{*\pm}\middle|z_R^\pm\right. + \alpha\beta^*\left\langle z_R^\pm\middle|z_R^{*\pm}\right\rangle = 2\,\mathrm{Re}\,\alpha^*\beta \tag{7.11}$$

Let us normalize these vectors as $2\,\mathrm{Re}\,\alpha^*\beta = 1$ (for a certain time $t = 0$ since, as we will see in a moment, this norm is not constant). Let us now define the projector

$$P_C^\pm = \left|z_C^\pm\right\rangle\left\langle z_C^\pm\right| \tag{7.12}$$

and compute

$$P_C^{\pm}|z_R^{\pm}\rangle = |z_C^{\pm}\rangle\langle z_C^{\pm}|z_R^{\pm}\rangle = |z_C^{\pm}\rangle(\alpha^*\langle z_R^{*\pm}| + \beta^*\langle z_{\bar{R}}^{\pm}|)|z_R^{\pm}\rangle = \alpha^*|z_C^{\pm}\rangle \tag{7.13}$$

From Eq. (5.3) we then see that

$$\begin{aligned} |z_C^{-}(t)\rangle &= (\alpha^*)^{-1}P_C^{-}|z_R^{-}(t)\rangle \\ &= (\alpha^*)^{-1}P_C^{-}e^{-iE_Rt}e^{-1/2\Gamma_Rt}|z_R^{-}(0)\rangle \\ &= e^{-iE_Rt}e^{-1/2\Gamma_Rt}|z_C^{-}(0)\rangle \end{aligned} \tag{7.14}$$

Thus

$$N = \langle z_C^{-}(t)|z_C^{-}(t)\rangle = e^{-\Gamma_Rt}\langle z_C^{-}(0)|z_C^{-}(0)\rangle = e^{-\Gamma_Rt} \tag{7.15}$$

Therefore N is a Lyapunov variable, and $|z_C^{-}(t)\rangle$ is a vector with a contractive evolution.

Therefore the formulation of resonances in quantum statistical mechanics requires a correct definition for resonance states. By correct, we mean free of inconsistencies. Nevertheless, we do not want to discuss this interesting problem here because it goes beyond the scope of the preset paper. In any case, one can see that our brackets as presented in the previous section may help in this direction

8. SCATTERING AND COSMOLOGICAL IRREVERSIBILITY

Finally, let us sketch the application of our formalism to two very important examples: scattering and cosmological models. In this way we shall understand the origin of time asymmetry. Let us begin with an scattering experiment, depicted in Fig. 4. A set of stable states $a_1, a_2, \ldots$ is transformed

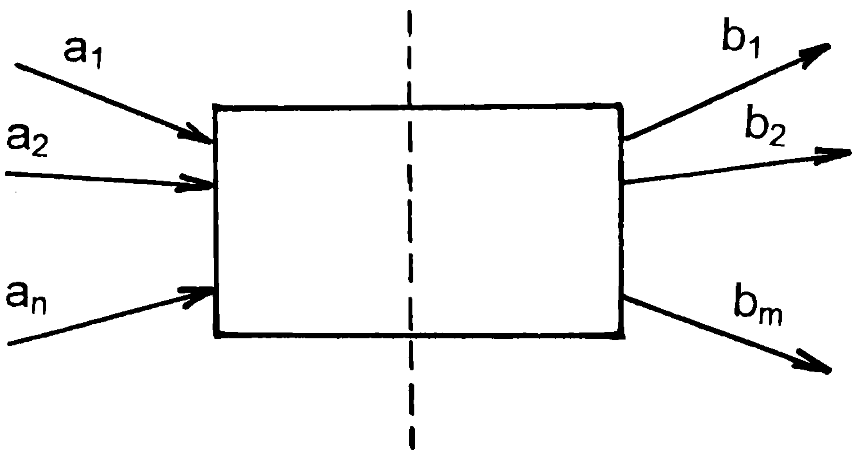

Fig. 4.

by the scattering process (the black box) in another set of stable states b_1, b_2, ...

This process can be looked at as reversible if we admit that there is only a conventional difference between "in" and "out" states, i.e., between past and future. There is no change of entropy in this process, as it is the usual *reversible* process that we know from ordinary quantum mechanics [This can be verified using the Y of (7.9) or N of Eq. (7.7) as the definition of the entropy.]

Now, let us cut the black box into two parts by the dotted line drawn in Fig. 4 at $t = 0$. Then, we can consider a first process, depicted in Fig. 5, where the stable states $a_1, a_2, \ldots$ creates the unstable states $u_1, u_2, \ldots$, which are growing states up to the time $t = 0$ and therefore belong to space $(\Phi^+)^\times$, like the Gamow vector $|z_R^{*+}\rangle$, Im $z_R^* > 0$. Therefore, this irreversible process will show a decrease of entropy (and this fact also can be verified by making the obvious changes in the definition of Y as the entropy). This would be the first half of Fig. 4, namely Fig. 5. We can also consider the second half of figure 4, namely Fig. 6, where some unstable states, belonging to the space $(\Phi^-)^\times$, decay after the time $t = 0$ into the stable states $b_1, b_2, \ldots$. These decaying states are like $|z_R^-\rangle$, Im $z_R > 0$. Therefore there must appear an increase of entropy, computed as in Eq. (7.9), in this irreversible process, which compensates the previous reduction of entropy in the creation process. Thus, we can see again that the "regular" physical Gamow vectors $|z_R^{*+}\rangle$ for $t > 0$ and $|z_R^-\rangle$ for $t > 0$ correspond to the growing and decaying real states, while the anomalous Gamow vectors $|z_R^{*-}\rangle$ and $|z_R^+\rangle$ permit us to extend the

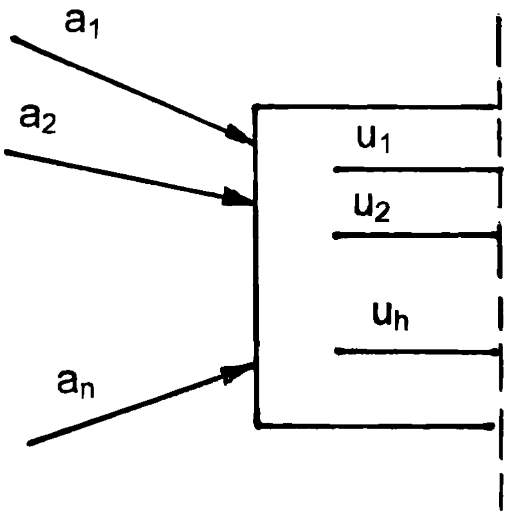

Fig. 5.

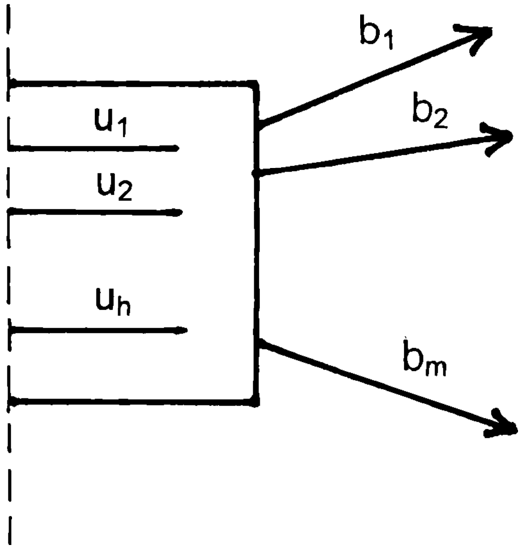

Fig. 6.

exponential time evolution of the physical Gamow vectors to $t > 0$ and $t < 0$, respectively.

The difference between Figs. 5 and 6 is just conventional, since one is just the mirror image of the other. However, while the process in Fig. 4 is reversible, the processes depicted in Figs. 5 and 6 are irreversible and there is a clear substantial difference between past and future in each of these figures. Should only decaying processes exist in the universe, we would choose Φ^- as our quantum space of states and use $(\Phi^-)^\times$ [and forget all about Φ^+ and $(\Phi^+)^\times$]. Thus, a substantial difference between past and future would be established in our model, which can be considered also as produced by the growing entropy. If the scattering were an isolated process, the choice between Φ^+ and Φ^-, or equivalently, the choice between decaying and growing states contained in $(\Phi^-)^\times$ and $(\Phi^+)^\times$, is really irrelevant since both situations are identical. In fact, life is the same in both models. We just call the future the direction of the decay of particles and the entropy growth, and the past, the opposite direction. But if the scattering process is not isolated, we can make a difference between preparation and measurement as in Section 5 and an arrow of time appears, although it is clear that it comes from the exterior of the system and not from the system itself.

The universe is a more complicated system than the one depicted in Fig. 4. Instead it is a system with a global growing entropy, as it starts with an initial decaying state at time $t = 0$ (the cut box). In fact, from (classical and quantum) cosmology we know that the universe began in an unstable state [42–45] as seen in Fig. 7, showing a *global branched scattering system* [46–48].

From the diagram we can see that all the arrows can be considered as states b_1, b_2, ... emerging from the corresponding boxes or from the half box in the far left of the diagram. Therefore they are all outgoing states contained in Φ^- and the corresponding unstable processes are decaying processes contained in $(\Phi^-)^\times$. This states are created in the corresponding box at a time t_0 and therefore they can only be extended to times $t \geq t_0$.

Of course, as in the previous case, we have the mirror image model for the universe (see Fig. 8) (with a global decrease of entropy in this case). Since the universe is isolated by definition, the choice between the models of figures 7 and 8 is irrelevant since both figures are really identical and life is the same in both models. Then, there is really no choice and time asymmetry appears as a consequence of the asymmetry of diagrams in Figs. 7 and 8. Today, we have only very elementary models for the universe showing all these characteristics [49]. In any case, we believe that they will show up in the future in more complete models. As in the scattering process depicted in Fig. 8, only regular Gamow vectors are used (as well as in Fig. 4) which represent the decaying processes within the universe and we see again that

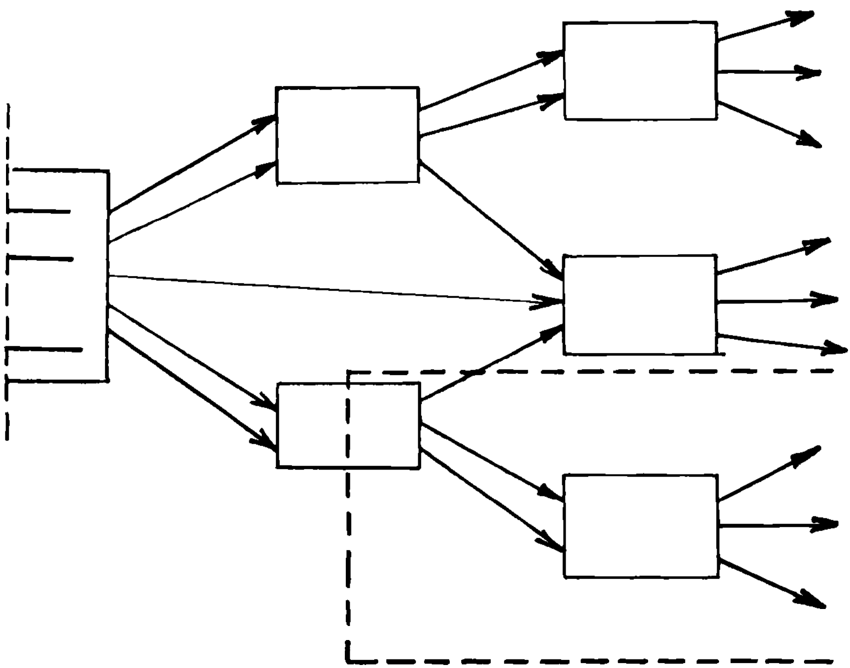

Fig. 7.

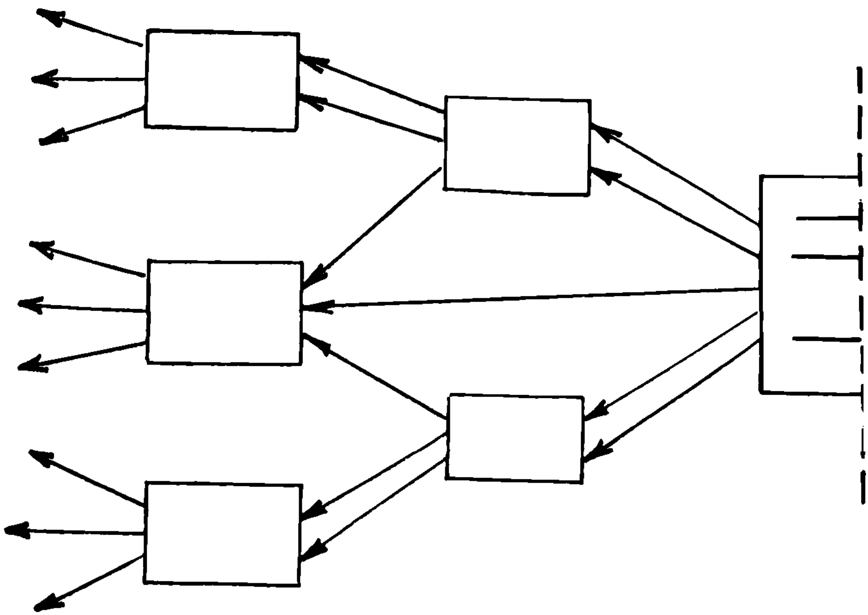

Fig. 8.

anomalous Gamow vectors do not appear in nature. They are only useful to pass from unitary evolution to nonunitary evolution and show the existence of contractive Lyapunov variables.

Therefore, in a cosmological model, we have reached the essence of the time asymmetry. It is impossible to break time symmetry using conventional quantum mechanics in Hilbert spaces. This symmetry can be broken only by finding in the theory two symmetrical structures and choosing one of these structures, taking into account that the choice is irrelevant.

Thus, we arbitrarily choose $\Gamma_+^\times$ or $\Gamma_-^\times$ (or $(\Phi^-)^\times$ or $(\Phi^+)^\times$) as our quantum state space for the universe (we have introduced these in this paper as examples of much more complicated cosmological models like the ones represented in Figs. 7 and 8). This choice is irrelevant since coincide with the choice between figures 7 and 8. If we choose Γ_-^x, following Section 4 we know that

$$K\colon \quad \Gamma_-^\times \mapsto \Gamma_+^\times \tag{8.1}$$

and therefore the time inversion K does not leave invariant the space of states Γ_-^x. Thus, physics in the space Γ_-^x is clearly irreversible. The same conclusion can be obtained by using the equivalent spaces $(\Phi^-)^\times$ and $(\Phi^+)^\times$ and the time-reversal operators $T_\pm$. The "master equation" of the theory can be obtained using the projectors P_C or a Λ-transformation at the density matrix level [50].

Once we have fixed the time direction in our cosmological model of the universe, we can better understand Figs. 5 and 6, which now can be interpreted just as elementary processes within the universe. It is the universe which fixes the global arrow of time and which tells the system where is the preparation and where is the measurement, i.e., its local quantum arrow of time. Thus, Fig. 5 depicts an open irreversible process, since it represents a system of particles that receives energy from some exterior source that accelerates the particles, represented by states a_1, a_2, . . . , and creates the unstable states u_1, u_2, ... This process is entropy-decreasing. Figure 6 represents a closed irreversible process, since now the energy is provided by the decay of the unstable states u_1, u_2, . . . and the stable states b_1, b_2, . . . are produced, with a growth of entropy.

Therefore, entropy grows in trivial irreversible closed systems. More generally, we can now prove that entropy does not grow only in the subsystems of the universe in which an irreversible process take place, for example, in systems like that represented by the dotted box of Fig. 7. The subsystem of Fig. 7 is nothing else than a scattering experiment plus a decay process. The latter provides the energy necessary to accelerate the ingoing particles. This can be described as the minimal model for a nontrivial closed system provided with its own energy source.

In every closed system, energy is necessary to produce reactions within the system and the only known way we have in the universe to obtain this energy it is to extract it from a decaying state (e.g., a battery where a chemical product decays into a more stable one, a star burning H, etc.). Therefore, in every closed system with one or several decaying states plus some scattering processes, the entropy grows and the second law of the thermodynamic holds. However, this makes sense if we have previously defined a global time direction for the whole universe only. If not, the second law would become meaningless.

Remark Although we must remember that the real experiment was the complete scattering experiment of Fig. 4, we ideally cut this experiment into two halves in Figs. 5 and 6 to show the nature of the growing state $|z_R^{*+}\rangle$ and the decaying state $|z_R^-\rangle$ (note that they are what we have called the "regular" Gamow states). As these processes are ideal, these states will never be isolated states in nature (with only one exception, as we shall see). They always appear together with the "continuous background." Then, while the scattering states are real physical states, the Gamow vectors $|z_R^-\rangle$ and $|z_R^{*+}\rangle$ always appear as component states and they are never isolated. Considering $|z_R^-\rangle$ as an isolated physical state is just a reasonable approximation if we can ignore the creation process of this unstable state, as would be the case if the lifetime of the unstable decaying states $|z_R^-\rangle$ were very large, e.g., the lifetime of some isotope, like C_{14}. In this sense even if our formalism in its "conceptual" version given in refs. 12, 33, and 46 can be considered as a way to introduce the time asymmetry, if we restrict ourselves to just the physics of the Gamow vectors, it turns out to be just an "effective theory" obtained when we neglect the preparation process.

However, there is one situation in which one is forced to ignore completely the creation process: when our point of departure is the initial state of the universe. Only in this cosmological situation can the unstable state represented by the Gamow vector $|\bar{z_R}\rangle$ be rigorously considered as a isolated physical state. Furthermore, only in this case do we find the initial state in the space $(\Phi^-)^\times$, since we shall consider the period $t > 0$ only, $t = 0$ being the *big bang* time. In fact, in ref. 40 it is shown that if the Hamiltonian of a system is bounded from below, there are no isolated states in Hilbert space. However, the Hamiltonian of the universe, the Wheeler–De Witt Hamiltonian, is the only one which is not bounded from below. According to this, the state of the universe would be the only isolated decaying state of nature. Also, the universe is completely isolated and is the only system where the arrow of time is produced by the system itself as here described.

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APPENDIX A

A rigged Hilbert space (RHS) is a triplet of spaces

$$\Phi \subset \mathcal{H} \subset \Phi^\times \tag{A.1}$$

with the following properties:

(i) $\mathcal{H}$ is a Hilbert space.

(ii) Φ is a dense subspace of $\mathcal{H}$ with its own complete nuclear topology. This topology on Φ is finer than the topology Φ has as subspace of $\mathcal{H}$. In particular, this means that the canonical embedding j: $\Phi \mapsto \mathcal{H}$ given by $j(\varphi) = \varphi$ for all φ in Φ is a continuous mapping. Nuclearity is a technical property that enables us to prove the Gel'fand–Maurin theorem to be presented below.

(iii) A functional on a given set is usually defined as a mapping from this set into the field of complex numbers. We say that F is an antilinear functional on Φ if, for any pair of complex numbers α, β and any pair φ, ψ in Φ, one has $F(\alpha\varphi + \beta\psi) = \alpha^* F(\varphi) + \beta^* F(\psi)$, where the star denotes complex conjugation. Then, $\Phi^\times$ is the vector space of all continuous antilinear functionals on Φ. $\Phi^\times$ has its own topology and $\mathcal{H}$ is dense in $\Phi^\times$ with respect to this topology. Also, the canonical mapping $j: \mathcal{H} \mapsto \Phi^\times$ is continuous.

Notation. Let $F \in \Phi^\times$ and $\varphi \in \Phi$. The action of F on φ is usually expressed as $F(\varphi)$, which is a complex number. Nevertheless, we prefer to use the notation $F(\varphi) = \langle\varphi|F\rangle$ in this paper. This has some adventages. First of all, it is consistent with the RHS implementation of the Dirac formulation of quantum mechanics [18–22]. This is also consistent with the fact that any h in the Hilbert space $\mathcal{H}$ also belongs to $\Phi^\times$. The action of h, as a member of $\Phi^\times$, on any φ in Φ is given by the scalar product $\langle\varphi|h\rangle$.

Let $\Phi \subset \mathcal{H} \subset \Phi^\times$ be a RHS and A an operator on $\mathcal{H}$ such that (i) Φ is contained in the domain (subspace of $\mathcal{H}$ in which A acts) of A. (ii) $A^\dagger\varphi \in \Phi$ for any φ in Φ, where $A^\dagger$ is the adjoint of A. This property is often expressed as $A^\dagger\Phi \subset \Phi$. (iii) $A^\dagger$ is continuous on Φ. Then, A can be extended to $\Phi^\times$, and this extension is a continuous (linear) operator on $\Phi^\times$, by means of the formula

$$\langle A^\dagger\varphi|F\rangle = \langle\varphi|AF\rangle, \qquad \forall\varphi \in \Phi, \quad \forall F \in \Phi^\times \tag{A.2}$$

It is customary to make a distinction between the operator A on $\mathcal{H}$ and its extension on $\Phi^\times$, usually denoted as $A^\times$. We do not make this distinction here for the sake of simplicity. It is obvious that if A were self-adjoint, formula (A.2) would read

$$\langle A\varphi|F\rangle = \langle\varphi|AF\rangle, \qquad \forall\varphi \in \Phi, \quad \forall F \in \Phi^\times \tag{A.3}$$

Formulas (A.2) and (A.3) define the action of A on any F in $\Phi^\times$.

Generalized Eigenvectors and Eigenvalues. Let A be an operator on $\mathcal{H}$ fulfilling conditions (i)–(iii) as above. An $F \in \Phi^\times$ is called a generalized eigenvector of A with eigenvalue λ if for any $\varphi \in \Phi$, one has

$$\langle A^\dagger\varphi|F\rangle = \langle\varphi|AF\rangle = \lambda\langle\varphi|F\rangle \tag{A.4}$$

for all φ in Φ. In this case, one writes $AF = \lambda\, F$ (or $A|F\rangle = \lambda|F\rangle$). Obviously, if A is self-adjoint, the dagger drops out in (A.4).

The next result is known as the nuclear spectral theorem or Gel'fand–Maurin theorem. We do not present it here with its full generality, although this version is sufficient for most cases of physical interest. The implementation of the Dirac formulation in terms of RHS mainly rests upon it.

Theorem (Gel'fand–Maurin). Let $A_1, A_2, \ldots, A_N$ be a complete set of commuting observables implemented as self-adjoint operators on a Hilbert space $\mathcal{H}$. We further assume that these operators have no singular spectrum. Then, there is a rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi^{\times}$ such that:

(i) The operators A_i leave Φ invariant, i.e., $A_i\Phi \subset \Phi$, $i = 1, 2, \ldots, N$. The A_i are continuous on Φ (they may not be continuous on $\mathcal{H}$) and therefore can be extended to a continuous operator on $\Phi^{\times}$.

(ii) Let $\sigma(A_i)$ be the Hilbert space spectrum of A_i and $X = \sigma(A_1) \times \ldots \times \sigma(A_N)$, the Cartesian product of these spectra. For any $(\alpha_1, \ldots, \alpha_N) \in X$, there is a vector $|\alpha_1, \ldots, \alpha_N\rangle$ in $\Phi^{\times}$, unique save for the product times a constant, fulfilling the following condition:

$$A_i|\alpha_1, \ldots, \alpha_N\rangle = \alpha_i|\alpha_1, \ldots, \alpha_N\rangle, \qquad i = 1, 2, \ldots, N \tag{A.5}$$

(iii) The set of generalized eiegenvectors $|\alpha_1, \ldots, \alpha_N\rangle$, as $(\alpha_1, \ldots, \alpha_N)$ runs over X, is complete in the sense that for any pair of vectors $\varphi, \psi \in \Phi$, one has a measure μ on X such that

$$(\varphi, \psi) = \int_X \langle\varphi|\alpha_1, \ldots, \alpha_N\rangle\langle\alpha_1, \ldots, \alpha_N|\psi\rangle \, d\mu(\alpha_1, \ldots, \alpha_N) \tag{A.6}$$

where, as is customary, $\langle\varphi|\alpha_1, \ldots, \alpha_N\rangle$ denotes the action of the functional $|\alpha_1, \ldots, \alpha_N\rangle \in \phi^{\times}$ on the vector $\varphi \in \Phi$ and $\langle\alpha_1, \ldots, \alpha_N|\varphi\rangle = \langle\varphi|\alpha_1, \ldots, \alpha_N\rangle^*$. As a consequence, if we omit the vectors φ and ψ, we can write

$$I = \int_X |\alpha_1, \ldots, \alpha_N\rangle\langle\alpha_1, \ldots, \alpha_N| \, d\mu(\alpha_1, \ldots, \alpha_N) \tag{A.7}$$

Also, for any function $f(\alpha_1, \ldots, \alpha_N)$ on X such that $f(A_1, \ldots, A_N)\,\Phi \subset \Phi$, one has the following spectral theorem, valid for any $\varphi, \psi \in \Phi$:

$$(\varphi, f(A_1, \ldots, A_N)\psi) = \int_X f(\alpha_1, \ldots, \alpha_N)\langle\varphi|\alpha_1, \ldots, \alpha_N\rangle\langle\alpha_1, \ldots, \alpha_N|\psi\rangle \, d\mu(\alpha_1, \ldots, \alpha_N) \tag{A.8}$$

This identity can also be written formally as

$$f(A_1, \ldots, A_N) = \int_X f(\alpha_1, \ldots, \alpha_N)|\alpha_1, \ldots, \alpha_N\rangle\langle\alpha_1, \ldots, \alpha_N| \, d\mu(\alpha_1, \ldots, \alpha_N) \tag{A.9}$$

As a corollary, one can show that for any self-adjoint operator A on a Hilbert space $\mathcal{H}$, one can find a RHS, $\Phi \subset \mathcal{H} \subset \Phi^{\times}$, such that:

(i) $A\Phi \subset \Phi^{\times}$ and A is continuous on Φ.

(ii) For any $\lambda \in \sigma(A)$, the Hilbert space spectrum of A, there exists $|\lambda\rangle \in \Phi^{\times}$, possibly degenerate, such that $A|\lambda\rangle = \lambda|\lambda\rangle$.

(iii) There exists a measure μ on $\sigma(A)$ such that

$$I = \int_{\sigma(A)} |\lambda\rangle\langle\lambda| \, d\mu(\lambda) \tag{A.10}$$

in the same sense as above. Analogously, if $f(A)\Phi \subset \Phi$, one has

$$f(A) = \int_{\sigma(A)} f(\lambda)|\lambda\rangle\langle\lambda| \, d\mu(\lambda) \tag{A.11}$$

It is important to remark that this Φ is not unique. In fact, if $\Psi \subset \mathcal{H} \subset \Psi^{\times}$ is another RHS for which $A\Psi \subset \Psi$ and A is continuous on Ψ, then (ii) and (iii) hold in $\Psi \subset \mathcal{H} \subset \Psi^{\times}$. This is what we sometimes call the *second version* of the *Gel'fand–Maurin* theorem.

We end this appendix with the following remark: The set of generalized eigenvalues of an operator on a given RHS may or may not coincide with its Hilbert space spectrum. We give here two examples:

(i) Consider the space S of the infinitely differentiable functions at all points that vanish at infinity faster than the inverse of any polynomial. The usual topology on S [27] makes the triplet $S \subset L^2(\mathbb{R}) \subset S^{\times}$ a RHS. Consider the momentum operator $P = -i\, d/dx$. The operator P leaves S invariant, is essentially self-adjoint on S, and is continuous on S. The set of its generalized eigenvalues coincides with the real line, hence with its Hilbert space spectrum.

(ii) Let $\mathcal{D}(\mathbb{R})$ be the space of all indefinitely differentiable functions at all points which vanish outside of a bounded interval. $\mathcal{D}(\mathbb{R})$ is endowed with a topology which makes $\mathcal{D}(\mathfrak{R}) \subset L^2(\mathbb{R}) \subset \mathcal{D}^{\times}(\mathbb{R})$ a RHS. The momentum operator P has on $\mathcal{D}(\mathbb{R})$ the same properties it has on S. Nevertheless, the set of its generalized eigenvalues coincides with the complex plane.

APPENDIX B. THE SPACES $\Delta_{\pm}$ AND $\Gamma_{\pm}$

The objective of the present appendix is to define the spaces $\Delta_{\pm}$ and $\Gamma_{\pm}$ as used in this paper. First of all, let us recall the definition of a Hardy function (or a Hardy class function) on a half–plane. An analytic function on the (open) upper half-plane $\mathbb{C}^{+} = \{z/\text{Im}\, z > 0\}$ is called a Hardy function if

$$\sup_{y>0} \int_{-\infty}^{\infty} |f(x+iy)|^2 \, dx = K < \infty \tag{B.1}$$

The definition of a Hardy class function on the lower half plane $\mathbb{C}^{-}$ is analogous. Boundary values of a Hardy function $f(z)$ on the real axis exist for almost (with respect to the Lebesgue measure) all points and determine a square-integrable function $f(x)$ that, reciprocally, uniquely determines all the values of $f(z)$ on the corresponding half-plane. For this reason, we identify a Hardy function with the function given by its boundary values on the real axis $\mathbb{R}$. Spaces of Hardy functions on $\mathbb{C}^{\pm}$ are called Hardy spaces and denoted by $\mathcal{H}^2_{\pm}$. For an account on the properties of Hardy functions, we refer the reader to the literature [51–53].

A Paley–Wiener theorem establishes that the Fourier transform $\mathcal{F}$ is a unitary mapping between the following couples of spaces [51, 54]:

$$\mathcal{F}: L^2(R^-) \mapsto \mathcal{H}^2_+$$
$$\mathcal{F}: L^2(R^+) \mapsto \mathcal{H}^2_- \tag{B.2}$$

Another result states that if $f(x)$ is a function which vanishes outside a bounded interval and admits derivatives at all points to all orders, its Fourier transform defines an entire analytic function [54].

Let us consider the vector space of all infinitely differentiable functions which are zero outside a compact interval contained in $\mathbb{R}^+$ ($\mathbb{R}^-$). We call this space $\mathcal{D}(\mathbb{R}^+)$ [$\mathcal{D}(\mathbb{R}^-)$]. Following the above-mentioned results, $\mathcal{F}(\mathcal{D}(\mathbb{R}^{\pm})) = \Delta_{\mp}$ is a space of functions which are

(i) Entire analytic.

(ii) Hardy functions on the $\left\{\substack{\text{upper}\\ \text{lower}}\right\}$ half-plane.

Let us consider the space $\mathcal{D}(\mathbb{R})$ as defined at the end of Appendix A. $\mathcal{D}(\mathbb{R}^{\pm})$ are closed subspaces of $\mathcal{D}(\mathbb{R})$ and therefore complete nuclear spaces [11]. Since the Fourier transform is a one-to-one mapping from $\mathcal{D}(\mathbb{R}^{\pm})$ onto $\Delta_{\mp}$, it also may transport the topologies from $\mathcal{D}(\mathbb{R}^{\pm})$ into $\Delta_{\mp}$. Henceforth, we will use these topologies for $\Delta_{\pm}$. These spaces determine two new RHS:

$$\Delta_{\pm} \subset \mathcal{H}^2_{\pm} \subset \Delta^{\times}_{\pm} \tag{B.3}$$

A theorem of van Winter [30] says that Hardy functions are determined by their boundary values on R^+. Let us consider $f_{\pm} \in \mathcal{H}^2_{\pm}$. Then, there exist one-to-one onto mappings $\theta_{\pm}$ such that they map $f_{\pm}(x)$ into their restrictions to $\mathbb{R}^+$, considered as different functions. By definition

$$\Gamma_{\pm} = \theta_{\pm}\, \Delta_{\pm} \tag{B.4}$$

This means that for any $g_{\pm} \in \Gamma_{\pm}$, there exists a unique $f_{\pm} \in \Delta_{\pm}$ such that $g_{\pm} = \theta_{\pm} f_{\pm}$ and, vice versa, for any $f_{\pm} \in \Delta_{\pm}$, one has $\theta_{\pm} f_{\pm} \in \Gamma_{\pm}$. The functions $\theta_{\pm}$ transport the topology from $\Delta_{\pm}$ into $\Gamma_{\pm}$, respectively. Further properites of $\Gamma_{\pm}$ [11, 30] enable us to establish that

$$\Gamma_{\pm} \subset L^2(R^+) \subset \Gamma^{\times}_{\pm} \tag{B.5}$$

are two new RHS. The operator multiplication $\mathcal{E}$ defined as $\mathcal{E} f(\omega) = \omega f(\omega)$

leaves $\Gamma_\pm$ invariant, is essentially self-adjoint on $\Gamma_\pm$, and is continuous in $\Gamma_\pm$. Therefore, it can be continuously extended to $\Gamma_\pm^\times$.

In summary, if $g(x)_\pm \in \Gamma_\pm$:

(i) It is a function defined on the positive part of the real axis $\mathbb{R}^+$.

(ii) It can be extended to an entire analytic function.

(iii) It is a Hardy function on the $\{\substack{\text{upper}\\\text{lower}}\}$ half-plane.

APPENDIX C

Our purpose in this appendix is to prove that a functional $F^\pm$ on $\Phi^\pm$ defined as $F^\pm(\phi^\pm) = [\phi^\pm(z^*)]^*$ is continuous for any $z \in \mathbb{C}$, $\mathbb{C}$ being the complex plane. Since the topologies on $\Phi^\pm$ and $\Delta_\mp$ are identical and the complex conjugation is always continuous, it suffices to prove that the mapping $\Delta_\pm \mapsto \mathbb{C}$ assigning to $f_\pm \in \Delta_\pm$ its value at z is continuous. Here, we prove the result on Δ_-, the proof for Δ_+ being identical.

Let $f(x) \in S$. We define

$$\|f\|_\infty = \sup_{x\in R} |f(x)|; \qquad \|f\|_2 = \sqrt{\int_{-\infty}^{\infty} |f(x)|^2\, dx} \tag{C.1}$$

One has the following result [27]:

$$\|f\|_\infty \leq \left\{ \left\|\frac{d}{dx} f\right\|_2 + \left\|x^2 \frac{d}{dx} f\right\|_2 \right\} = C\{p_{01}(f) + p_{21}(f)\} \tag{C.2}$$

where

$$p_{nm}(f) = \left\| x^n \frac{d^m}{dx^m} f \right\|_2 \tag{C.3}$$

is a family of seminorms defining the topology on S [27].

Now, let $f \in \Delta_-$ and $z \in \mathbb{C}$. Following the definition of Δ_- given in Appendix A, there must exist a $\varphi(k) \in S$ supported on $[0, b]$ such that for $z \in \mathbb{C}$

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikz} \varphi(k)\, dk = \frac{1}{\sqrt{2\pi}} \int_0^b e^{-ikz} \varphi(k)\, dk \tag{C.4}$$

Taking moduli in (C.4) and using (C.2), one has

$$|f(z)| \leq C \sup_{k\in R} |\varphi(k)| = C\|\varphi\|_\infty \leq C'\{p_{01}(\varphi) + p_{21}(\varphi)\} \tag{C.5}$$

The Plancherel theorem says that [27]

$$\left\| k^n \frac{d^m}{dk^m} \varphi \right\|_2 = \left\| \frac{d^n}{dx^n} x^m f \right\|_2 \tag{C.6}$$

Thus,

$$|f(z)| \leq C'\{p_{10}(f) + 2p_{01}(f) + p_{12}(f)\} \tag{C.7}$$

Since the topology on Δ_- is a strict inductive limit of the topologies on the spaces of Fourier transforms of functions supported on the intervals of the form $[0, b_i]$, $b_1 < b_2 < \ldots b_i < \ldots \mapsto \infty$, the Dieudonné Schwartz theorems [55] prove the continuity of our functional.

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